A relative fundamental lemma for U (4)

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ABSTRACT

In (HLR86), Harder et al. presented a proof of the Tate conjecture, an important conjecture in the field of arithmetic geometry, for the non-CM part of the cohomology of Hilbert modular surfaces. In this thesis, we present a general strategy for a study of the Tate conjecture for some unitary Shimura varieties. As in the work cited above, we do this by studying the notion of *distinction*. Distinction on a unitary group is related to distinction on a general linear group through a comparison of relative trace formulas. In the latter setting, work of Jacquet and his collaborators has led to simple criteria in terms of base change and L-functions for the existence of distinguished representations of GL_N. The main result of this thesis is then a proof of a special case of a relative fundamental lemma, the first ingredient of the comparison, when the unitary group is of rank 4.

ABRÉGÉ

Dans leur article (HLR86), Harder et al. présente une preuve de la conjecture de Tate, une importante conjecture en géométrie arithmétique, pour la partie sans CM de la cohomologie des surface modulaires de Hilbert. Cette thèse propose une stratégie générale pour l'étude de la conjecture de Tate pour certaines variétés de Shimura unitaires. Comme dans le travail cité ci-haut, la méthode proposée passe par l'étude de la notion de *distinction*. La distinction sur un groupe unitaire est reliée à la distinction sur un groupe général linéaire par le biais d'une comparaison de formules des traces relatives. Dans ce dernier contexte, le travail de Jacquet et ses collaborateurs donne des critères simples, en termes de changement de base et de fonctions L, pour l'existence de représentations distinguées sur GL_N. Le résultat principal de cette thèse est donc une preuve d'un cas spécial d'un lemme fondamental relatif, premier ingrédient d'une comparaison, lorsque le groupe unitaire est de rang 4.

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Introduction

Arithmetic geometry can be defined as the study of geometric objects defined over interesting (from a number-theoretic point of view) rings. For instance, étale cohomology allows us to package some geometric information about a variety into a more manageable, linear-algebraic object. Moreover, and this was the main reason for introducing ℓ-adic cohomology theories in the 1950's, some *arithmetic* information as well is retained in these cohomology groups, in the form of a *Galois representation*. One example of the interplay between geometry and arithmetic is the celebrated *Tate conjecture*, that was first introduced in (Tat65) and that we now briefly present. Let V be a geometrically irreducible smooth projective variety of dimension d over a field k. We assume that k is finitely generated over its prime subfield, and we write $\Gamma = \mathrm{Gal}(\overline{k}/k)$ for its absolute Galois group. We fix a prime ℓ and write

$$H^n(V) := H^n_{\text{ét}}(V_{\overline{k}}, \mathbb{Q}_\ell).$$

As mentioned above, these cohomology groups are equipped with an action of Γ, and we let $H^n(V)(j)$ be the j-th Tate twist. If X is an irreducible subvariety of codimension i, Poincaré duality allows us to assign a cohomology class $c(X) \in H^{2i}(V)(i)$, which can be extended by additivity to what is called the (i-th) *cycle map* (recall that an i-th cycle is a $\mathbb{Z}$-linear combination of irreducible subvarieties of codimension

i). We will let $Z^i(V)$ denote the subgroup generated by those irreducible subvarieties which are defined over the base field k.

Conjecture 0.0.1 (Tate)**.** *The image of $Z^i(V)$ under the cycle map generates the subgroup of Galois-invariant cohomology classes in $H^{2i}(V)(i)$, that is*

$$c(Z^i(V)) \otimes \mathbb{Q}_\ell = H^{2i}(V)(i)^\Gamma.$$

Moreover, the dimension of $Z^i(V)$ is equal to the order of the pole at $s = i$ of the Hasse-Weil zeta-function of V.

This conjecture is currently very much open, although important special cases have been proven. For example, it has been proven for the non-CM part of the cohomology of Hilbert modular surfaces (HLR86) (when $i = 1$). The technique used for this special case is the main motivation for this thesis, and we will explain the ingredients that come into it.

Let F be a real quadratic number field, and set $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_2)$. Let V be the Shimura variety associated to G and a congruence subgroup $K \subset G(\mathbb{A}_f)$, where $\mathbb{A}_f$ is the finite part of the adele ring $\mathbb{A}$ of $\mathbb{Q}$; these varieties are *Hilbert modular surfaces.* In this setting, the group $c\,(Z^1(V))$ is reasonably well-understood: it is generated by the Hirzebruch-Zagier cycles, studied in (HZ76). On the cohomology side, the action of the Hecke algebra commutes with the action of the Galois group, and we get the following decomposition:

$$H^{2i}(V) = \bigoplus \pi_f^K \otimes X^i(\pi),$$

where $\pi = \pi_f \otimes \pi^\infty$ runs through automorphic representations of $G(\mathbb{A})$, and where $X^i(\pi)$ is a finite-dimensional Galois representation. The representations appearing in the decomposition have been studied by Brylinski and Labesse in (BL84), building on earlier work of Langlands. Since the Hecke algebra also acts on $Z^1(V)$, we obtain a similar decomposition. Therefore, the study of the Tate conjecture can be done by studying individual automorphic representations.

Definition 0.0.2. Let G be a connected reductive group over a number field F, and let $H \subset G$ be a closed, connected reductive subgroup. A cuspidal representation π of $G(\mathbb{A}_F)$ is said to be H-*distinguished* if

$$\mathcal{P}(\phi) := \int_{H(F)\backslash H(\mathbb{A}_F)\cap {}^1G(\mathbb{A}_F)} \phi(g)dg$$

is nonzero for some cusp form in the π-isotypic subspace of $L^2(G(F)\backslash {}^1G(\mathbb{A}_F))$ (here, ${}^1G(\mathbb{A}_F)$ denotes the Harish-Chandra subgroup of $G(\mathbb{A}_F)$, defined in §2.2 below).

The authors then proved that the representations appearing in the decomposition of $Z^1(V)$ are precisely the representations distinguished by $\mathrm{GL}_{2/\mathbb{Q}}$, and they also relate the period integrals $\mathcal{P}(\phi)$ to the poles of the Hasse-Weil zeta function of V.

This result was quickly generalized to inner forms of GL_2 (Lai85) using a *relative trace formula*. However, when $n > 2$, the locally symmetric space associated to GL_n is not hermitian. In particular, they do not give rise to Shimura varieties, and therefore the same techniques do not directly yield other cases of the Tate conjecture.

In this thesis, we wish to present a strategy that can ultimately lead to a generalization of the work of Harder et al. for some unitary Shimura varieties. Consider the following setting: let M/F be a CM extension of number fields, and set $G = \mathrm{Res}_{M/F}(\mathrm{GL}_{2n})$. Let H be a unitary group in $2n$ variables over M/F, realized as the subgroup of fixed points of an involution τ on G. We can choose a second involution σ on G such that $G^\sigma = \mathrm{Res}_{M/F}(\mathrm{GL}_n) \times \mathrm{Res}_{M/F}(\mathrm{GL}_n)$ and H^σ is a product of two copies of a unitary group in n variables. As noted above, the proof in (HLR86) consists of a good understanding of both the image of the cycle map and of the automorphic representations appearing in the decomposition of $H^{2i}(V)(i)^\Gamma$. It is expected (although not thoroughly understood yet) that H^σ-distinguished representations of $H(\mathbb{A}_F)$ should give rise to cohomologically non-trivial cycles on the unitary Shimura variety associated to H (or a suitable generalization). On the other hand, the Galois theory of the cohomology of Shimura varieties is reasonably well-understood (for an overview of this topic, see (BR94)), and in particular, the work of Jacquet and Friedberg provides a link to distinction, in terms of Asai L-functions. The strategy we propose is the following: to understand Tate's conjecture for unitary Shimura varieties, we wish to study distinction on H. We propose to do this by relating it to the study of distinction on G, via a *comparison of trace formulas*: we want to compare a relative trace formula on H with a *twisted* relative trace formula on G. This comparison can ultimately lead us to a sufficient condition for a cuspidal automorphic representation of H to be distinguished by H^σ. By relating the study of distinction on a unitary group to that on a general linear group, we can now make use of the work of Jacquet and his collaborators, which has led to the following characterization:

Theorem 0.0.3 ((FJ93), (Jac10))**.** *A cuspidal representation* Π *of* $G(\mathbb{A}_F)$ *is distinguished by* G^σ *if and only if the (partial) exterior square L-function attached to* Π *has a pole at* $s = 1$ *and* $L^S(\frac{1}{2}, \Pi) \neq 0$ *(for a carefully chosen finite set of primes* S*). Moreover,* Π *is distinguished by a quasi-split unitary group if and only if* Π *is* τ*-invariant.* □

The spectral side of the twisted relative trace formula on G isolates representations which are both G^σ- and G^θ-distinguished, where $\theta = \sigma \circ \tau$. One can easily show that G^θ is a quasi-split unitary group. By Theorem 0.0.3, a cuspidal representation of GL_n is distinguished by G^θ if and only if it comes from base change. Therefore, using Theorem 2.2.2 of Harris and Labesse (HL04), we can start with a cuspidal representation π of H and consider its weak base change Π to G; it will thus automatically be G^θ-distinguished. The comparison of trace formulas we present in this thesis should eventually lead us to a proof of the following conjecture.

Conjecture 0.0.4. *Let* F *be a number field and let* G *and* H *be as above. Let* π *be a cuspidal automorphic representation of* $H(\mathbb{A}_F)$*, and suppose there exists a cuspidal automorphic representation* Π *of* $G(\mathbb{A}_F)$ *that is a weak base change of* π*. Let* S *be a finite set of primes containing the infinite places and all places where* Π *is ramified. If* $L^S(s, \wedge^2\Pi)$ *has a pole at* $s = 1$ *and* $L^S(\frac{1}{2}, \Pi) \neq 0$*, then there is a* H^σ*-distinguished cuspidal automorphic representation* π' *nearly equivalent to* π*.*

This has been conjectured by Getz and Wambach (GW). Note that it may be necessary to make additional assumptions on the central characters or incorporate character twists.

In this thesis, we do not carry out the full trace comparison. However, we present the different tools essential to this setting, and we prove a special case of the relevant relative fundamental lemma. The author is hopeful that a full proof of the fundamental lemma can be achieved through more geometric means, and he hopes to carry out the computations in the near future, as well as pursue the program presented above.

* * *

This thesis is organized as follows: in the first chapter, we introduce the basic vocabulary pertaining to algebraic groups. We explain some constructions that will be useful in later chapters and we give a classification theorem for unitary groups. In the second chapter, we give a brief overview of Langlands' functoriality and explain how the Arthur-Selberg trace formula can be used to provide a correspondence between automorphic representations of "related" reductive groups. We also explain the parallels between this trace formula and Jacquet's relative trace formula, which is the computational tool that allows us to relate *distinguished* representations on two groups. The last two chapters give an introduction to the first steps in the comparison of relative trace formulas: we provide a matching between certain classes on a general linear group and those on a unitary group, and then use this to prove a special case of a relative fundamental lemma. We also explain in what way a comparison of relative trace formulas is relevant to our situation.

All global and local fields appearing in this thesis are assumed to be of characteristic zero. Therefore, global fields are number fields, and local fields are finite extensions of an archimedean or non-archimedean completion of $\mathbb{Q}$.

CHAPTER 1
Algebraic Groups

In this chapter, we set up the vocabulary that will be used throughout this thesis. We give the definition of group schemes over a general ring (unless otherwise stated, all rings are assumed commutative and with identity), although we will mostly be interested in group schemes over fields and local rings. We also cover some general constructions that will be used later. We then present the construction of unitary groups and state their basic properties.

1.1 Group schemes - Main definitions

Let k be a ring. An *(affine) group scheme* over k is a representable functor G from the category of k-algebras to the category of groups. If need be, the representing algebra will be denoted by A_G. If this algebra is finitely-generated, we say G is an *algebraic group scheme*; if k is a field and $A_G \otimes_k \overline{k}$ is reduced, we say G is an *algebraic group*. We also note that, equivalently, a group scheme over k is an affine scheme over $\mathrm{Spec}(k)$ such that its functor of points factors through the forgetful functor from the category of groups to that of sets (see (EH00, Theorem IV-14) for a characterization of functors coming from schemes). This justifies the name, and also, all constructions available to schemes are available to group schemes (e.g. fiber products).

Definition 1.1.1. Let $H \to G$ be a morphism of group schemes over k. We say H is a *closed subgroup* of G if $H(R) \subset G(R)$ is a subgroup (in the usual sense) for all k-algebra R.

By Yoneda's lemma, this implies that H is representable by a quotient of A_G, hence the adjective *closed*.

Example. Let R denote a k-algebra.

- Define

$$\mathbb{G}_m(R) = R^\times.$$

  This functor can be represented by the k-algebra $A_{\mathbb{G}_m} = k[X, Y]/(XY - 1)$. We will call this algebraic group scheme the *multiplicative group*.

- More generally, define

$$\mathrm{GL}_n(R) = \{g \in \mathrm{Mat}_n(R) : \det(g) \neq 0\}.$$

  This functor can be represented by the k-algebra

$$A_{\mathrm{GL}_n} = k[X_{11}, \ldots, X_{nn}, Y]/(det(X_{11}, \ldots, X_{nn})Y - 1).$$

  We will call this algebraic group scheme the *general linear group*. Note that $\mathrm{GL}_1 = \mathbb{G}_m$.

- Let Γ be a finite group. Define

$$\Gamma(R) = \mathrm{Maps}(R, \Gamma),$$

  that is, $\Gamma(R)$ is the set of all functions from R to Γ and the group structure is given by pointwise multiplication. This functor is representable by the k-algebra $A_\Gamma = \mathrm{Maps}(k, \Gamma)$ and is called the *constant group scheme*.

For another important class of examples, see § 1.2.

A construction that will play an important role in what follows is the *restriction of scalars* (à la Weil). This is to be construed as an inverse operation to the more familiar operation of base change.

Definition 1.1.2. (CGP10, §A.5) Let $k \to k'$ be a finite flat map of Noetherian rings, and let X' be an affine scheme over k'. The *Weil restriction* of X' along $k \to k'$, denoted $\mathrm{Res}_{k'/k}(X')$, is the unique (affine) scheme of finite type over k satisfying the universal property

$$\mathrm{Res}_{k'/k}(X')(R) = X'(R \otimes_k k')$$

for all k-algebras R.

The existence of this scheme is proven in (BLR90, §7.6). We note that if X' is a group scheme, so is $\mathrm{Res}_{k'/k}(X')$ (CGP10, Propositions A.5.1). Since restriction of scalars commutes with base change (CGP10, Proposition A.5.2), we have the following result.

Proposition 1.1.3. *Suppose k is a perfect field, and let k'/k be a finite separable extension. Fix an extension K/k containing all translates of k', and let G be a group scheme over k'. Then*

$$(\mathrm{Res}_{k'/k}(G))_K \cong \prod_{\sigma:k'\to K} G^\sigma.$$

□

1.1.1 Integral models

For this section, we fix a non-archimedean local field F, with ring of integers $\mathcal{O}$. Let $\mathfrak{m} = (\pi)$ be its maximal ideal, and let $\kappa = \mathcal{O}/\mathfrak{m}$.

Definition 1.1.4. Let X be a scheme of finite type over F. An *integral model* of X is a scheme $\mathcal{X} = \mathrm{Spec}(A)$ over $\mathcal{O}$ such that $A \subset A_X$ is a sub-$\mathcal{O}$-algebra of finite type (over $\mathcal{O}$) and such that $A[1/\pi] = A_X$. We will say the model is *smooth* if the morphism $\mathcal{X} \to \mathrm{Spec}(\mathcal{O})$ is smooth; we say it is *connected* if both fibers are connected.

A consequence of this definition is that a model $\mathcal{X}$ of X is flat and has generic fiber X. We also note that the restriction of scalars of a model is a model for the restriction of scalars, and that this operation preserves both smoothness and connectedness (Yu, §2.5).

Let G be a finite group acting on a model $\mathcal{X}$ of X by automorphisms over $\mathcal{O}$. Then $\mathcal{X}^G$ is affine with generic fiber X^G, and it represents the functor $R \to \mathcal{X}(R)^G$ on the category of $\mathcal{O}$-algebras. In this context, we have the following result:

Proposition 1.1.5. *(Edi92, §3.4) If $\mathcal{X}$ is smooth and $\#G$ is invertible in $\mathcal{O}$, then $\mathcal{X}^G$ is a smooth model of X^G.* □

We will need the following very useful cohomological result.

Lemma 1.1.6. *Suppose G is a smooth connected group scheme over a Henselian (local) ring k with finite residue field. Then every flat G-torsor is trivial. In particular, $G(k)$ is non-empty.*

Proof. Since G is smooth, (Str83, Theorem 1) implies that G-torsors are in bijection with torsors over the special fibre. Since G is connected, these are all trivial by Lang's theorem (Lan56, Theorem 2). The last assertion is then a trivial consequence of the definitions (see (Mil80, §III.4)). □

When we discuss the fundamental lemma in later chapters, we will be interested in a specific model that we now introduce. Let G be a reductive group over F. The group of F-points is naturally endowed with a topology coming from that of F. Let

$K \subset G(F)$ be a maximal compact subgroup. We say it is a *hyperspecial subgroup* if there exists a smooth and connected integral model $\mathcal{G}$ of G such that $\mathcal{G}(\mathcal{O}) = K$. We have the following important existence result.

Proposition 1.1.7. *(Mil92, §1, citing (Tit79), 1.10.2) There exists a hyperspecial subgroup $K \subset G(F)$ if and only if G is unramified, that is, G is quasi-split and it splits over an unramified extension of F.* □

Example. Consider $G = \mathrm{GL}_n$. If we look at the obvious model $\mathcal{G} = \mathrm{GL}_n$, we see that both fibers are connected, and so this is a smooth and connected integral model for G. We want to show that $K = \mathcal{G}(\mathcal{O}) = \mathrm{GL}_n(\mathcal{O})$ is a maximal compact subgroup of $G(F)$. First of all, we note that since K is the intersection of a compact subset $(\mathrm{Mat}_n(\mathcal{O}))$ and a closed subset $(\det^{-1}(\mathcal{O}^\times))$ of $\mathrm{Mat}_n(F)$, it is compact. These two sets are also open, and therefore so is K. Also, K is the stabilizer of the standard lattice $\mathcal{O}^n$ in F^n. It is thus enough to prove that any compact subgroup stabilizes a lattice, since any such subgroup can then be conjugated to lie inside K. Let $C \subset G(F)$ be a compact subgroup. Since K is open, $K \cap C$ is open in C and so $C/(C \cap K)$ is finite. If we set

$$L = \sum_{g \in C/(C \cap K)} g\mathcal{O}^n,$$

then L is a lattice, being a finite sum of lattices. But clearly C stabilizes L, and the result follows.

1.2 Unitary groups

Let F be any field of characteristic different from 2, and let M be a two-dimensional étale algebra over F. Denote the non-trivial F-automorphism of M

by c (we will also use the notation $\overline{x} := c(x)$). Let V be a free M-module of rank n, and let

$$q : V \times V \to M$$

be a *hermitian form*, i.e. q is c-linear in the first variable, linear in the second variable, and for all $x, y \in V$ we have $q(y, x) = c(q(x, y))$. We also assume that q is *non-degenerate*, that is, for all nonzero $x \in V$, there exists $y \in V$ such that $q(x, y) \neq 0$. We can attach to this data an algebraic group in the following way.

Definition 1.2.1. The *unitary group* H attached to the data (F, M, V, q) introduced above is the algebraic group over F whose functor of points is given by

$$H(R) = \{g \in \mathrm{GL}(V \otimes_F R) : q(gx, gy) = q(x, y), \text{ for all } x, y \in V \otimes_F R\},$$

where R is an F-algebra.

Proposition 1.2.2. *The functor H is an algebraic group.*

Proof. In view of Proposition 1.2.4 below, the only non-trivial case is when M/F is a quadratic extension of fields. In that case, let $e \in M \setminus F$. This element satisfies a quadratic polynomial over F, and by completing the square (recall that $\mathrm{char}(F) \neq 2$), we can assume that $e^2 \in F$ and $\overline{e} = -e$. Then any matrix with coefficients in $M \otimes_F R$ can be written as $A + eB$, where A and B are matrices with coefficients in R. The condition that $A + eB \in H(R)$ is then

$$(A + eB)(A^t - eB^t) = I,$$

which is equivalent to

$$AA^t - e^2BB^t = I$$
$$AB^t - BA^t = 0.$$

Therefore, H is representable by a quotient of $F[X_{11}, \ldots, X_{nn}] \otimes_F F[Y_{11}, \ldots, Y_{nn}]$. $\square$

To understand when two quadruples (F, M, V, q) and (F, M, V', q') give rise to isomorphic unitary groups, we have the following result.

Lemma/Definition 1.2.3. We say that two hermitian spaces (V, q) and (V', q') are *equivalent* if there exists an M-linear isomorphism $\phi : V \to V'$ such that $q'(\phi(x), \phi(y)) = q(x, y)$ for all $x, y \in V$. We say that (V, q) and (V', q') are *quasi-equivalent* if (V, q) is equivalent to $(V', \alpha q')$ for some $\alpha \in M^\times$. If (V, q) and (V', q') are quasi-equivalent, then they give rise to F-isomorphic unitary groups.

Proof. First, we simply note that if H_α denotes the unitary group associated to $(V, \alpha q)$, then $H(R) = H_\alpha(R)$ for all F-algebras R. In particular, they are isomorphic.

Now, suppose (V, q) and (V', q') are equivalent, and let $\phi : V \to V'$ be an isomorphism which proves this. We will denote the algebraic group corresponding to (V, q) by H, and the algebraic group corresponding to (V', q') by H'. We will also denote the trivial extension of ϕ to $V \otimes_F R$ by ϕ. Consider the following map:

$$H(R) \to H'(R)$$
$$g \mapsto \phi g \phi^{-1}.$$

This is well-defined: let $g \in H(R)$. Then we have

$$q'(\phi g \phi^{-1}(x), \phi g \phi^{-1}(y)) = q(g\phi^{-1}(x), g\phi^{-1}(y)) = q(\phi^{-1}(x), \phi^{-1}(y)) = q(x, y),$$

where the last equality follows from the fact that the inverse of an isometry is an isometry. This map is also clearly an invertible group homomorphism. Therefore, the result follows. □

Given a basis $e_1, \ldots, e_n$ of V, the determinant of the matrix $(q(e_i, e_j))_{i,j}$ is independent of the chosen basis when viewed as an element of $F^\times/\mathrm{Nm}_{M/F}(M^\times)$; this element is called the *discriminant* of the hermitian space V.

Example. We will classify the low dimensional hermitian spaces over local fields. Let M/F be a quadratic extension of non-archimedean local fields. We say $v \in V$ is *isotropic* if $v \neq 0$ and $q(v, v) = 0$. A subspace $U \subset V$ is called *anisotropic* if $q(u, u) \neq 0$ for all $u \in U$; it is called *totally isotropic* if every vector is isotropic.

1. Suppose $n = 1$, and let $\alpha \in F^\times$. We define the hermitian space $V = M(\alpha)$ by setting

   $$q(e, e') = \bar{e}\alpha e'.$$

   Clearly, every one-dimensional hermitian space has this form (we can recover $\alpha = q(1, 1)$), and they are all quasi-equivalent. We note that $q(e, e) \in F^\times/\mathrm{Nm}_{M/F}(M^\times)$ is independent of $e \in M$ (in fact, it is exactly the coset of α).

2. Suppose $n = 2$, and suppose there exists an isotropic vector $v \in V$. Since q is non-degenerate, there exists a nonzero $w \in V$ such that $q(v, w) \neq 0$. Let

$\alpha \in F$, and consider

$$q(w + \alpha v, w + \alpha v) = q(w, w) + \mathrm{Tr}_{M/F}(\alpha(q(v, w)).$$

Since the trace is surjective, we can force this to be zero by choosing α appropriately. Finally, by rescaling the resulting vector, we see that we can choose w such that $q(w, w) = 0$ and $q(v, w) = 1$. We will call such a totally isotropic hermitian space a *hyperbolic plane* and denote it $\mathcal{H}$. The discriminant is easily seen to be -1.

3. We can also construct an anisotropic two-dimensional hermitian space as follows. Let $\alpha, \beta \in F^\times$ and consider $V = M(\alpha) \oplus M(\beta)$. The hermitian form is given by

$$q((e_1, e_2), (e_1', e_2')) = \overline{e_1}\alpha e_1' + \overline{e_2}\beta e_2'.$$

Therefore, V is anisotropic if and only if $-\alpha/\beta \notin \mathrm{Nm}_{M/F}(M^\times)$.

In the definition of unitary groups, we have allowed M to be any two-dimensional étale algebra. Therefore, either M/F is a quadratic extension of fields, or $M \cong F \times F$. In the latter case, unitary groups are a very familiar object.

Proposition 1.2.4. *Suppose $M \cong F \times F$. Then the unitary group associated to (F, M, V, q) is F-isomorphic to* GL_n.

Proof. Choose an F-morphism $p : M \to F$, and let $V_0 = V \otimes_{M,p} F$. Then, we can define an isomorphism $H(R) \cong \mathrm{GL}(V_0 \otimes_F R)$ by sending $g \in H(R)$ to the M-automorphism it induces on $(V \otimes_M R) \otimes_{M,p} F = V_0 \otimes_F R$. □

Fix the data (F, M, V, q) and its unitary group H, and let F' be a field extension of F. Then $M' := M \otimes_F F'$ is still an étale algebra. Moreover, we can consider $V' := V \otimes_M M'$ (which is still free of rank n over M') and q', the natural extension of q to V'. We thus get a quadruple (F', M', V', q'), to which we can associate an F'-algebraic group. Clearly, this algebraic group is the base change $H_{F'}$. We are led to the following observations:

1. If M is a field and we take $F' = M$, then $M' = M \times M$, and so the base change H_M is isomorphic to GL_n. Therefore, unitary groups are always forms of GL_n.
2. If M/F is an extension of number fields and v is a place of F which splits in M, then H_{F_v} is again isomorphic to GL_n.

1.2.1 A Classification Theorem

We now want to give a classification of unitary groups over local non-archimedean fields.

Theorem 1.2.5. *Let M/F be a quadratic extension of local non-archimedean fields. If n is odd, there is a unique isomorphism class of unitary groups; these groups are quasi-split. If $n = 2m$ is even, then there are two isomorphism classes of unitary groups, only one of which is quasi-split.*

The proof of the above theorem will occupy the remainder of this section. First, we present the classification of hermitian spaces up to quasi-equivalence.

Lemma 1.2.6. *Let V be an n-dimensional hermitian space.*

1. *If $n = 2m + 1$ is odd, then V is isometric to the sum of m copies of $\mathcal{H}$ and $M(\alpha)$, for some $\alpha \in F^\times$.*

2. If $n = 2m$ is even, then V is isometric to the sum of m copies of $\mathcal{H}$ or to the sum of $m-1$ copies of $\mathcal{H}$ and $M(\alpha) \oplus M(\beta)$, where $-\alpha/\beta \notin \mathrm{Nm}_{M/F}(M^\times)$.

Therefore, if n is odd, there exists only one hermitian space, up to quasi-equivalence. If n is even, there are two hermitian spaces, up to quasi-equivalence.

Proof. First, we note that since $q(v, v) \in F$ for all $v \in V$, we can define a quadratic form $Q(v) := q(v, v)$ on the $2n$-dimensional F-vector space V. By (Ser73, Chapter IV.2, Theorem 6), the quadratic space V has an isotropic vector whenever $2n \geq 5$. Therefore, the hermitian space V has an isotropic vector whenever $n \geq 3$. Working recursively, we see that any hermitian space can be decomposed into a sum of hyperbolic planes and a non-trivial subspace of dimension at most two. The result now follows from our classification above. □

From this, we know that there is one isomorphism class when n is odd and at most two when n is even. The number of hyperbolic planes appearing in the decomposition of V is called the *Witt index*.

Proposition 1.2.7. *The dimension of a maximal split torus in H is equal to the Witt index of the corresponding hermitian space.*

Proof. We will denote the maximal anisotropic subspace of V by V_0. Let m be the Witt index of V. We fix a basis of V, respecting the decomposition coming from Lemma 1.2.6, as follows: for the i-th hyperbolic plane, we choose isotropic vectors $\{e_i, e'_i\}$ such that $q(e_i, e'_i) = 1$, and we complete this to a basis of V by choosing a basis for V_0. We define the subgroup

$$A(R) = \{(\lambda_1, \lambda_1^{-1}, \ldots, \lambda_m, \lambda_m^{-1}, 1, \ldots, 1) : \lambda_i \in R^\times\}.$$

This representation is taken with respect to our fixed basis. Clearly, we have $A \cong (\mathbb{G}_m)^m$ over F: it is a split torus. We claim it is maximal. Indeed, if S is a torus containing A, it fixes the maximal anisotropic subspace V_0 of V (since V_0 is the eigenspace corresponding to the eigenvalue 1 for the elements of A). Therefore, if S is split, its restriction to V_0 is again a split torus of the unitary group attached to V_0. But the unitary group of an anisotropic hermitian space is itself anisotropic, and so the restriction of S to V_0 is trivial. Therefore $S = A$.

□

As a consequence, we deduce that unitary groups are non-split, and so they are *outer* forms of GL_n.

To finish the proof of Theorem 1.2.5, we let A be the maximal split torus defined above, and we let T be the centralizer of A. First, suppose $V = \mathcal{H}$. We have

$$T(R) = \{(\lambda, \overline{\lambda}^{-1} : \lambda \in R^\times\}$$

(to centralize, the matrix has to be diagonal; the fact that it lies in the unitary group forces the form above). Thus $T \cong \mathrm{Res}_{M/F}\mathbb{G}_m$. For the general case, we first suppose V is isometric to the sum of m copies of the hyperbolic plane. We then have

$$T \cong (\mathrm{Res}_{M/F}\mathbb{G}_m)^m$$
$$A \cong (\mathbb{G}_m)^m.$$

Similarly, suppose V is of odd-dimension. We then have

$$T \cong (\mathrm{Res}_{M/F}\mathbb{G}_m)^m \times U(1)$$
$$A \cong (\mathbb{G}_m)^m,$$

where $U(1)$ is the unitary group associated to the anisotropic line. Finally, suppose V has a two-dimensional anisotropic subspace. We then have

$$T \cong (\mathrm{Res}_{M/F}\mathbb{G}_m)^m \times U(V_0)$$
$$A \cong (\mathbb{G}_m)^m,$$

where $U(V_0)$ is the unitary group associated to the anisotropic two-dimensional hermitian space. Therefore, we note that when n is odd or V has no anisotropic subspace, T is a maximal torus (since it is commutative). By (Spr98, Proposition 16.2.2), this implies that the corresponding unitary group is quasi-split. Conversely, when n is even and V has a two-dimensional anisotropic subspace, T is not a maximal torus (since it is not commutative), and so the same Proposition implies that the corresponding unitary group is *not* quasi-split.

Example. Let V be a hermitian space of even dimension n. In this case, determining if the corresponding unitary group is quasi-split can be seen from the discriminant: it is quasi-split if and only if the discriminant is congruent to $(-1)^{n/2}$ modulo $\mathrm{Nm}_{M/F}(M^\times)$. We will investigate two particular hermitian spaces: let $V = M^n$, and define

$$q(v,v) = \overline{v}^t \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix} v$$

$$q'(v,v) = \overline{v}^t \begin{pmatrix} I_{2n} & \\ & -I_{2n} \end{pmatrix} \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix} v.$$

By choosing the standard basis of M^n, we deduce that in both cases, the discriminant is equal to $(-1)^n = 1$. Therefore, the corresponding unitary groups are quasi-split if and only if $n \equiv 0 \pmod 4$.

CHAPTER 2
Functoriality and Trace Formulas

Langlands' *Principle of functoriality* has been a major unifying force in number theory for the past forty years. It is within this frame that most of the work on the trace formula in the past thirty years has been done. Early on, it was apparent that the comparison of trace formulas on two different reductive groups could allow us to give a correspondence between their respective automorphic representations. Hence, our introduction to trace formulas will follow the same motivation. We first give a very brief introduction to functoriality, mainly focusing on examples relevant to our discussion. We then explain how the trace formula can be used to prove instances of functoriality, and we give an introduction to the relative trace formula, which is the main tool we will use.

2.1 Langlands' Philosophy and Functoriality

The classification of connected reductive groups over a non-algebraically closed field F consists of two pieces of data:

- The (absolute) root datum coming from the base change to an algebraic closure;
- Descent data in terms of an action of the absolute Galois group.

Using this, Langlands defined a *dual* group, called the *L-group*, as follows. Let G be a connected redutive group over a local or global field F. Taking the dual root datum gives rise to another reductive group $G^\vee$ over $\mathbb{C}$, and the identity component of the L-group is the group of complex points of this reductive group; it is a complex

Lie group. Then, the action of the absolute Galois group of F on the root datum induces an action on the dual root datum, and we can thus set

$$^LG := G^\vee(\mathbb{C}) \rtimes \mathrm{Gal}(\overline{F}/F);$$

the action on $G^\vee(\mathbb{C})$ is defined via a choice of *épinglage* (for more detail, see (Cog04, §1)). Note that the L-group determines the group G up to inner twists (and so uniquely defines a quasi-split group). As a somewhat trivial, yet important, example, we note that $^LG = G^\vee(\mathbb{C}) \times \mathrm{Gal}(\overline{F}/F)$ if and only if G is a split group.

Remark. It is sometimes convenient to replace the L-group above by its *Weil form*, i.e. we replace the absolute Galois group by the absolute Weil group (Tat79).

Langlands' principle of functoriality states that homomorphisms from a conjectural *Langlands group* into the L-group should somehow parametrize the automorphic (if F is global) or admissible (if F is local) representations of G, and as such, L-homomorphisms $^LG \to {}^LH$ should translate to a correspondence between automorphic or admissible representations of G and those of H. Here, an *L-homomorphism* is a continuous map between the L-groups that restricts to the identity map on the Galois group and whose restriction to the identity component is a complex-analytic homomorphism.

We now present two important examples that will help illustrate these concepts.

2.1.1 Base change

Let G be a connected reductive group over a number field F, and let E/F be a finite extension of degree m. Set $H = \mathrm{Res}_{E/F}(G_E)$. At the level of L-groups, the connected component of ^LH is m copies of the connected component of ^LG, and so we have a natural L-homomorphism ${}^LG \to {}^LH$: it is the diagonal embedding on the identity component and the identity on Galois groups. The conjectural instance of functoriality corresponding to this map is known as *base change*.

Some cases have been proven when $G = \mathrm{GL}_n$. When $n = 2$ and E/F is cyclic of prime degree, it was proven by Saito, Shintani and Langlands (Lan80). For $n > 2$, this is a theorem of Arthur and Clozel. Since we will need this result later on, we record it here. The extension E/F is still cyclic of prime degree; fix a generator ξ of the Galois group.

Theorem 2.1.1. *(AC89, Theorem 3.4.2) Let π be a cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Assume $\pi \not\cong \pi \otimes \eta$, where η is the character associated to E/F by (global) class field theory. Then there exists a unique ξ-stable cuspidal representation Π of $\mathrm{GL}_n(\mathbb{A}_E)$ lifting π. Conversely, let Π be a cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_E)$. If $\Pi \cong \Pi^\xi$, then there exists a cuspidal representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ lifting to Π.* □

Without giving too many details, we will simply note that "being a lift" is defined in terms of L-functions (the interested reader is referred to (AC89, §3.1)). Blasius and Rogawski have also investigated the base change for unitary groups in two and three variables (BR92). However, almost nothing is known in general about nonsolvable base change, which would allow one to prove the Artin conjecture in the so-called icosahedral case (Get12, Corollary 1.9).

2.1.2 Unitary groups

Let E/F be a quadratic extension of number fields. Let H be a quasi-split unitary group in n variables with respect to this extension and set $G = \mathrm{Res}_{E/F}(\mathrm{GL}_n)$.

Since H is a form of GL_n, we have

$$^{L}H = \mathrm{GL}_n(\mathbb{C}) \rtimes \mathrm{Gal}(\overline{F}/F).$$

We note that this action is *not* trivial (because H is non-split), and so ^{L}H is not isomorphic to $^{L}\mathrm{GL}_n$. We can again define an L-homomorphism

$$^{L}H \to {}^{L}G = \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \times \mathrm{Gal}(\overline{F}/F)$$

by taking the diagonal embedding on the identity component and the identity map on Galois groups.

In this situation, we have the following theorem of Harris and Labesse. Here τ is the automorphism of G fixing H.

Theorem 2.1.2. *(HL04, Theorem 2.2.2) Assume all archimedean places are complex. Let* Π *be a cuspidal representation of* $H(\mathbb{A}_F)$ *which is locally supercuspidal at two split places. Then there exists a cuspidal representation* π *of* $G(\mathbb{A}_F)$ *which is* τ*-stable and which is the weak base change of* Π. □

The local restriction appearing in the hypothesis is due to the lack of a stable trace formula at the time (HL04) was published. However, this problem has now been resolved due to the work of many people (most notably Laumon and Ngo in the final stages), and recent work of Mok (Mok) should allow us to remove this local restriction.

* * *

One important aspect these two examples of functoriality share is that they were proven using a comparison of trace formulas, which leads us to our next topic.

2.2 Arthur-Selberg Trace Formula

We fix a connected reductive group G over a number field F.

2.2.1 Harish-Chandra subgroup

Let A_G be the largest central subgroup of $\mathrm{Res}_{F/\mathbb{Q}}G$ over $\mathbb{Q}$ that is a $\mathbb{Q}$-split torus; let k be the rank of A_G. The identity component $A_G(\mathbb{R})^\circ$ of $A_G(\mathbb{R})$ (with respect to the real topology) is isomorphic to k copies of $(\mathbb{R}^\times)^\circ$, and so it is also isomorphic to the additive group $\mathbb{R}^k$. Let $X(G)_\mathbb{Q}$ be the group of $\mathbb{Q}$-rational characters of G; this is a free abelian group of rank k. We define a morphism

$$HC_G : G(\mathbb{A}_F) \to \mathrm{Hom}(X(G)_\mathbb{Q}, \mathbb{R})$$

by

$$\langle HC_G(x), \chi \rangle = \left| \log \left(\chi(x) \right) \right|, \qquad x \in G(\mathbb{A}_F), \chi \in X(G)_\mathbb{Q}.$$

This map is surjective, and we define the *Harish-Chandra subgroup* of $G(\mathbb{A}_F)$ by

$$^1G(\mathbb{A}_F) := \ker(HC_G).$$

Note that $G(\mathbb{A}_F) = {}^1G(\mathbb{A}_F) \times A_G(\mathbb{R})^\circ$ and that $G(F) \subset {}^1G(\mathbb{A}_F)$.

Example. Let $G = \mathrm{GL}_n$. Then $A_G \cong \mathrm{GL}_1$, and so $X(G)_\mathbb{Q}$ is isomorphic to $\mathbb{Z}$; a canonical generator is given by the determinant. Therefore, we have

$$^1\mathrm{GL}_n(\mathbb{A}_F) = \{x \in \mathrm{GL}_n(\mathbb{A}_F) : \det(x) = 1\}.$$

We also note that the quotient $G(F)\backslash{}^1G(\mathbb{A}_F)$ has finite volume (Art05, §3).

2.2.2 Spectral and Geometric decomposition

To streamline the discussion and minimize the number of inaccuracies, we will henceforth assume that the derived subgroup of G is simply connected. This hypothesis can be removed at the cost of more complicated definitions. We note that all groups appearing in Chapter 3 and 4 satisfy this hypothesis.

Consider the right regular representation of $G(\mathbb{A}_F)$ on $L^2(G(F)\backslash {}^1G(\mathbb{A}_F))$. This action gives rise to an action of $C_c^\infty(G(\mathbb{A}_F))$ by integration:

$$(R(f)\phi)(x) := \int_{^1G(\mathbb{A}_F)} f(y)\phi(xy)dy, \qquad \phi \in L^2(G(F)\backslash {}^1G(\mathbb{A}_F)),$$

where $f \in C_c^\infty(G(\mathbb{A}_F))$ and dy is a choice of Haar measure on ${}^1G(\mathbb{A}_F)$. If we make a change of variables and use the definition of quotient measure on $G(F)\backslash {}^1G(\mathbb{A}_F)$, we get

$$(R(f)\phi)(x) = \int_{G(F)\backslash {}^1G(\mathbb{A}_F)} \left(\sum_{\gamma \in G(F)} f(x^{-1}\gamma y) \right) \phi(y)dy;$$

since f is compactly supported and $G(F)$ is discrete, the sum is finite. Therefore, $R(f)$ is an integral operator, with kernel

$$K_f(x,y) := \sum_{\gamma \in G(F)} f(x^{-1}\gamma y).$$

The first goal of the trace formula is to give a spectral and a geometric decomposition of this kernel. The theory of Eisenstein series gives an explicit decomposition

$$L^2(G(F)\backslash {}^1G(\mathbb{A}_F)) = \bigoplus_{\chi} L^2_{\chi},$$

where $\chi = (M, \sigma)$ is a *cuspidal datum*: M is the Levi component of a parabolic subgroup of G and σ is a cuspidal representation of $M(\mathbb{A}_F)$ (Art05, §12). Accordingly, we get a decomposition

$$K_f(x, y) = \sum_{\chi} K_{f,\chi}(x, y).$$

However, the component $K_{f,\chi}$ is not necessarily integrable over the diagonal. For this reason, Arthur introduced his *truncation operator* (of which we will say nothing; see (Art05, §13)) to modify the kernel and deal with divergence issues. In any case, $K_{f,\chi}$ will be integrable if $M = G$. Let

$$L_0^2(G(F)\backslash{}^1G(\mathbb{A}_F)) = \bigoplus_{\chi=(G,\pi)} L_\chi^2;$$

this is the *cuspidal part* of $L^2(G(F)\backslash{}^1G(\mathbb{A}_F))$. We have the following theorem:

Theorem 2.2.1. *(GGPS69, Chapter 3.7) The space $L_0^2(G(F)\backslash{}^1G(\mathbb{A}_F))$ decomposes into a discrete sum of irreducible representations with finite multiplicities.* □

From this, we deduce the spectral decomposition (for the cuspidal part): let $R_0(f)$ be the restriction of $R(f)$ to the cuspidal part of the spectrum, and similarly for $K_f(x, y)$ and $R(g)$. The above theorem tells us that

$$R_0(g) \cong \sum_{\pi} m_\pi \pi(g),$$

and we thus have

$$\begin{aligned}\mathrm{Tr} R_0(f) &= \int_{G(F)\backslash{}^1G(\mathbb{A}_F)} K_{f,0}(x, x)dx \\ &= \sum_{\chi=(G,\pi)} \int_{G(F)\backslash{}^1G(\mathbb{A}_F)} K_{f,\chi}(x, x)dx.\end{aligned}$$

Moreover, we have

$$K_{f,\chi}(x,y) = \sum_{\phi} (R(f)\phi)(x)\overline{\phi}(y),$$

where $\{\phi\}$ is an orthonormal basis for the π-isotopic subspace of $L_0^2(G(F)\backslash {}^1G(\mathbb{A}_F))$.

On the geometric side, we can decompose our kernel as a sum over conjugacy classes $\mathcal{O}$:

$$K_f(x,y) = \sum_{\mathcal{O}} K_{f,\mathcal{O}}(x,y),$$

where $K_{f,\mathcal{O}}(x,y) = \sum_{\gamma\in\mathcal{O}} f(x^{-1}\gamma y)$. Again, the component $K_{f,\mathcal{O}}(x,y)$ is not necessarily integrable over the diagonal, and we still need to use truncation operators. Nonetheless, it will be integrable if the conjugacy class $\mathcal{O}$ is elliptic, and we have the following result. For a group G and an element $\gamma \in G(F)$, let G_γ be the centralizer of γ in G.

Proposition 2.2.2. *(Gel96, Proposition 2.6) If $\mathcal{O}$ is elliptic, we have*

$$\int_{G(F)\backslash {}^1G(\mathbb{A}_F)} K_{f,\mathcal{O}}(x,x)dx = \mathrm{vol}(G_\gamma(F)\backslash G_\gamma(\mathbb{A}_F)) \int_{G_\gamma(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} f(x^{-1}\gamma x)dx,$$

where $\gamma \in \mathcal{O}$. The integral appearing on the right is the orbital integral *of γ at f.* □

The (unrefined) Arthur-Selberg trace formula is then an equality of sums of distributions:

$$\sum_{\mathcal{O}} J_{\mathcal{O}}^T(f) = \sum_{\chi} J_{\chi}^T(f), \qquad f \in C_c^\infty(G(\mathbb{A}_F)).$$

Here T is a truncation parameter, which is trivial when $\mathcal{O}$ is elliptic and $\chi = (G, \pi)$, respectively. Also, from our discussion above, we have

$$J^T_{\mathcal{O}}(f) = \mathrm{vol}(G_\gamma(F)\backslash G_\gamma(\mathbb{A}_F)) \int_{G_\gamma(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} f(x^{-1}\gamma x)dx, \qquad \text{when } \mathcal{O} \text{ is elliptic,}$$

$$J^T_\chi(f) = \int_{G(F)\backslash G(\mathbb{A}_F)} \left(\sum_\phi (R(f)\phi)(x)\overline{\phi}(x) \right) dx, \qquad \text{when } \chi = (G, \pi).$$

Remark. Although the coefficients appearing in the trace formula (the volumes and multiplicities) are global, the orbital integrals are *local* objects. Indeed, the measure on the group of adelic points factors as a product of measures on the groups of local points, and therefore, if $f = \otimes_v f_v$ is factorizable, then the global orbital integral factorizes as a product of local orbital integrals:

$$\int_{G_\gamma(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} f(x^{-1}\gamma x)dx = \prod_v \int_{G_{\gamma_v}(F_v)\backslash G(F_v)} f_v(x^{-1}\gamma_v x)dx.$$

2.2.3 Twisted Trace Formula

The above formalism can also be applied to *twisted groups*: let θ be an automorphism of G and consider the (disconnected) group $G \rtimes \langle\theta\rangle$; the connected component indexed by θ is called a twisted group. The trace formula we obtain (called the *twisted trace formula*) is different from the original formula in two aspects: on the spectral sides, the representations appearing are exactly the θ-invariant automorphic representations of G; on the geometric side, conjugacy classes are replaced by θ-conjugacy classes. As is apparent in Theorems 2.1.1 and 2.1.2, the image of the transfer map is characterized by being invariant under an automorphism of the group (the Galois automorphism in the former, an outer automorphism of GL_n in the latter). Approaching these instances of functoriality via a comparison of a twisted trace formula and a trace formula is therefore natural.

2.2.4 Comparison of trace formulas

Now that we have a formula relating the cuspidal representations of a reductive group G to geometric data, we could try to match the conjugacy classes in $G(F)$ with those of $H(F)$, for a second reductive group H. If this is possible, then we get a relation between the geometric sides of the trace formula, which gives us a relation between the spectral sides. We could then try to make it explicit by choosing appropriate test functions. However, this is not possible in general: we can only hope for a correspondence between geometric classes (i.e. conjugacy classes over the algebraic closure). The discrepancy between conjugacy and geometric conjugacy is encapsulated in the notion of *stability* (for an informal introduction, see (Art97)).

Kottwitz, using ideas of Langlands, first gave a general construction that should lead to a *stabilization* of the trace formula (a decomposition of both sides in terms of distributions invariant under stable conjugacy). More precisely, assuming some technical lemmas, he was able to stabilize the cuspidal part of the spectral side and the elliptic part of the geometric side of the formula. At the moment, the stablization of the cuspidal part is still open, but recent work of Ngo on the fundamental lemma has led to an unconditional stablization of the elliptic part. (For a very readable introduction to the stable trace formula, see (Har11).)

Once the conjugacy classes have been matched, there is also the issue of matching the *test functions* on each group. This can be done locally, and at almost all places, we can restrict ourselves to bi-K-invariant functions, where K is a carefully chosen compact subgroup of $G(F_v)$. By doing so, this space of functions is now an algebra (called the *Hecke algbera*) and we can ask that our matching of functions be an algebra homomorphism. This is the point of view we will adopt in Chapter 3 and 4.

In turns out that this method of endoscopic comparison has serious limitations with respect to proving functoriality (for example, it is powerless with respect to symmetric powers lifting); it should be enough only for functoriality arising from "small" L-homomorphisms (to paraphrase Ngo), i.e. when ${}^L H$ is the subgroup of fixed point of an involution on ${}^L G$ and the L-map is the inclusion. Still, this formalism is enough to obtain a full classification of cuspidal representations of classical quasi-split groups in terms of representations of GL_n (see (Art13) for orthogonal and symplectic groups and (Mok) for unitary groups).

2.3 Jacquet's Relative Trace Formula

The relative trace formula was introduced in (JL85) by Jacquet and Lai to study the notion of distinction and ultimately cycles on algebraic varieties. At its essence, it is a tool that allows one to study harmonic analysis on symmetric spaces. We will therefore try to present it as an extension of the absolute trace formula; we follow the exposition in (Jac97).

Let F and G be as above, and fix an involution ϵ on G. Let S be the F-subscheme of G whose points in an F-algebra R are given by

$$S(R) = \{s \in G(R) : s^\epsilon = s\};$$

it is a subvariety of G, and we assume it is connected. We have a right action of $G(F)$ on $S(F)$ defined by $g \cdot s := gsg^{-\epsilon}$; we assume that this action is transitive. Fix a point $\omega \in S(F)$, and let H be its stabilizer. Suppose that $\phi \in C_c^\infty(S(\mathbb{A}_F))$, and define

$$K_\phi(g) := \sum_{s \in S(F)} \phi(gsg^{-\epsilon}), \qquad g \in G(\mathbb{A}_F).$$

This function is well-defined (the sum is finite), and it is invariant under left translation by $G(F)$. Let $\mathcal{K}$ be the space spanned by the functions K_ϕ. Note that this space is invariant under right translations by $G(\mathbb{A}_F)$, and so we can try to decompose this representation as in the absolute case.

First of all, we can choose $f \in C_c^\infty(G(\mathbb{A}_F))$ such that f is K_∞-finite, where K_∞ is a maximal compact subgroup of $G(F_\infty)$, and

$$\phi(s) = \int_{H(F)\backslash H(\mathbb{A}_F) \cap {}^1G(\mathbb{A}_F)} f(gh)dh, \qquad s = g\omega g^{-\epsilon},$$

We can consider the (absolute) kernel K_f as above and take its spectral decomposition. This translates to a spectral expansion for K_ϕ:

$$K_\phi = \sum_\chi K_{\phi,\chi}, \tag{2.3.0.1}$$

where

$$K_{\phi,\chi}(g) = \int_{H(F)\backslash H(\mathbb{A}_F) \cap {}^1G(\mathbb{A}_F)} K_{f,\chi}(h,g)dh.$$

As above, χ is a cuspidal data. When $\chi = \{G, \pi\}$, we have

$$K_{\phi,\chi}(g) = \sum_i \mathcal{P}(\rho(f)\psi_i)\overline{\psi_i}(g),$$

where $\{\psi_i\}$ is an orthonormal basis of the π-isotypic subspace of $\mathcal{K}$ and

$$\mathcal{P}(\phi) := \int_{H(F)\backslash H(\mathbb{A}_F)\cap^1 G(\mathbb{A}_F)} \phi(h)dh.$$

Therefore, $K_{\phi,\chi}$ is nonzero if and only π is distinguished by H.

Just as in the absolute case, we can also decompose K_ϕ *geometrically*:

$$K_\phi = \sum_{s\in S(F)/H(F)} K_{\phi,s}. \tag{2.3.0.2}$$

When s is elliptic regular semisimple, $K_{\phi,s}$ is integrable over the diagonal, and we have

$$\int_{H(F)\backslash H(\mathbb{A}_F)\cap^1 G(\mathbb{A}_F)} K_{\phi,s}(x,x)dx = \mathrm{vol}(H_s(F)\backslash H_s(\mathbb{A}_F)) \int_{H_s(\mathbb{A}_F)\backslash H(\mathbb{A}_F)\cap^1 G(\mathbb{A}_F)} \phi(hsh^{-\epsilon})dh.$$

As above, the *relative trace formula* is the equality of expansions 2.3.0.1 and 2.3.0.2.

As in the absolute case, there exists a *twisted* variant, introduced in (Hah09) and (GW). This formula is analogous to the Deligne-Kazhdan simple trace formula: we need to impose strong local conditions in order to kill the divergent integrals on both sides. And as in the absolute case, there are also problems arising from the "instability" of the original trace formula. A prestabilization has been obtained by Getz and Wambach (GW, Proposition 7.2); a full stabilization is at the moment out of reach and will probably require the introduction of new ideas. In particular, there is currently no conjectural framework for a theory of relative endoscopy. Investigating examples on which to base a theory of relative endoscopy is one of the primary aims of this thesis.

CHAPTER 3
Relative classes

We now enter the main part of this thesis. As explained in the introduction, a comparison of relative trace formulas should allow us to study distinction on a unitary group in terms of distinction on a general linear group. In this chapter, we carry the first step of this comparison: we provide a matching between the relative classes appearing in the geometric side of each formula. In what follows, R denotes an F-algebra.

3.1 Twisted and untwisted relative classes

Let F be a local or global field (recall our convention about the characteristic of such fields), and let M/F be a quadratic extension. We will denote by $x \mapsto \overline{x}$ the non-trivial automorphism of M/F. We fix an algebraic closure $\overline{F}$ of F containing M and denote by Γ the absolute Galois group of F. Set $G := \mathrm{Res}_{M/F}\mathrm{GL}_{4n}$ (cf. Chapter 1), for $n \geq 1$. If ϵ is an involution on G, let G^ϵ be the subgroup of fixed points of ϵ, that is, the subgroup of G whose points in an F-algebra R are given by

$$G^\epsilon(R) = \{g \in G(R) : g^\epsilon = g\}.$$

An element $\delta \in G(R)$ is said to be *ϵ-split* if $\delta^\epsilon = \delta^{-1}$. More generally, a torus is *ϵ-split* if all its elements are ϵ-split. We will consider three different involutions on

G: first set

$$J = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}, \qquad I_{2n,2n} = \begin{pmatrix} I_{2n} & \\ & -I_{2n} \end{pmatrix} \in G(F).$$

Now define $g^\tau := J\overline{g}^{-t}J$, and let σ be the involution given by conjugation by $I_{2n,2n}$. Finally, set $\theta := \sigma \circ \tau$; we note that σ and τ commute. Since the group G^τ will play a special role in the sequel, we set $H := G^\tau$. The groups H and G^θ are unitary groups, and by the computations at the end of Chapter 1, they are both quasi-split.

Consider the following (left) actions:

$$\begin{aligned} (H^\sigma \times H^\sigma) \times H(R) &\to H(R) \\ (h_1, h_2, h) &\mapsto h_1 h h_2^{-1} \end{aligned}$$

and

$$\begin{aligned} (G^\sigma \times G^\theta) \times G(R) &\to G(R) \\ (g_1, g_2, g) &\mapsto g_1 g g_2^{-1}. \end{aligned}$$

The double cosets under these actions can be related to conjugacy classes in the following way. We follow (Ric82, Lemma 2.4) and introduce the subschemes $Q \subset H$ and $S \subset G$; they are respectively defined as the scheme-theoretic image of the following moment maps:

$$B_\sigma : H(R) \to H(R)$$
$$h \mapsto hh^{-\sigma}$$
$$B_\theta : G(R) \to G(R)$$
$$g \mapsto gg^{-\theta}.$$

As closed subschemes of affine schemes, both Q and S are themselves affine. Pertaining to the above actions, we note the following relation: $B_\sigma(h_1 h h_2^{-1}) = h_1 B_\sigma(h) h_1^{-1}$ and $B_\theta(g_1 g g_2^{-1}) = g_1 B_\theta(g) g_1^{-\tau}$. By definition, the moment maps induce isomorphisms:

$$H^\sigma \backslash H / H^\sigma \to H^\sigma \backslash Q$$
$$G^\sigma \backslash G / G^\sigma \to G^\sigma \backslash S;$$

here H^σ acts by conjugation on Q and G^σ, by τ-conjugation on S.

Definition 3.1.1. We will say two elements $\delta, \delta_0 \in G(F)$ lie in the same *twisted relative class* if they have the same image in $G^\sigma(F) \backslash S(F)$. Similarly, two elements $\gamma, \gamma_0 \in H(F)$ lie in the same *relative class* if they have the same image in $H^\sigma(F) \backslash Q(F)$.

The comparison of relative trace formulas we want to perform relies on a matching between the relative classes and the twisted relative classes.

Now, consider the following subgroup schemes:

$$H_\gamma(R) = \{(h_1, h_2) \in H^\sigma \times H^\sigma(R) : h_1 \gamma h_2^{-1} = \gamma\}$$
$$G_\delta(R) = \{(g_1, g_2) \in G^\sigma \times G^\theta(R) : g_1 \delta g_2^{-1} = \delta\},$$

where $\gamma \in H(F)$ and $\delta \in G(F)$. Denote by C_{γ,H^σ} (resp. C^τ_{δ,G^σ}) the centralizer of γ in H^σ (resp. the τ-centralizer of δ in G^σ). We have the following lemma.

Lemma 3.1.2. *Projection onto the first factor induces isomorphisms*

$$H_\gamma \cong C_{\gamma\gamma^{-\sigma},H^\sigma} \qquad \textit{and} \qquad G_\delta \cong C^\tau_{\delta\delta^{-\theta},G^\sigma}.$$

Proof. Let R be an F-algebra. We first check injectivity. Suppose $(g_1, g_2), (g_1, \tilde{g}_2) \in G_\delta(R)$. Then, we have

$$g_1 \delta g_2^{-1} = \delta = g_1 \delta \tilde{g}_2^{-1},$$

and so $g_2 = \tilde{g}_2$. A similiar argument works, *mutatis mutandis*, for H_γ.

Now, suppose $g \in G^\sigma(R)$ is such that $g\delta\delta^{-\theta}g^{-\tau} = \delta\delta^{-\theta}$ (or equivalently, $\delta^{-\theta} g^\tau \delta^\theta = \delta^{-1} g \delta$). Then

$$(\delta^{-1} g \delta)^\theta = \delta^{-\theta} g^\tau \delta^\theta = \delta^{-1} g \delta,$$

and $g\delta(\delta^{-1}g\delta)^{-1} = \delta$. Therefore, $(g, \delta^{-1}g\delta) \in G_\delta(R)$ is a preimage of g. The same argument shows that if $h \in H^\sigma(R)$ is such that $h\gamma\gamma^{-\sigma}h^{-1} = \gamma\gamma^{-\sigma}$, then $(h, \gamma h \gamma^{-1}) \in H_\gamma(R)$ is a preimage of h. □

As in the absolute case, we have a notion of *stability*.

Definition 3.1.3. We say that two relatively semisimple elements $\gamma, \gamma_0 \in H(F)$ are in the same *stable relative class* if $x\gamma\gamma^{-\sigma}x^{-1} = \gamma_0\gamma_0^{-\sigma}$ for some $x \in H^\sigma(\overline{F})$. Similarly, two relatively semisimple elements $\delta, \delta_0 \in G(F)$ are in the same *stable relative τ-class* if $x\delta\delta^{-\theta}x^{-\tau} = \delta_0\delta_0^{-\theta}$ for some $x \in G^\sigma(\overline{F})$.

From the above definition, we see that if γ and γ_0 are in the same stable relative class, then we have an inner twist

$$H_\gamma \cong C_{\gamma\gamma^{-\sigma},H^\sigma} \cong C_{\gamma_0\gamma_0^{-\sigma},H^\sigma} \cong H_{\gamma_0}$$
$$h \mapsto xhx^{-1},$$

where x is such that $x\gamma\gamma^{-\sigma}x^{-1} = \gamma_0\gamma_0^{-\sigma}$. Similarly, we have an inner twist $G_\delta \cong G_{\delta_0}$ whenever δ and δ_0 are in the same stable relative τ-class.

As in the absolute case, we may have a disparity between relative classes (resp. relative τ-classes) and stable relative classes (resp. stable relative τ-classes). To study this discrepancy, we make the following definition. For a pair of connected reductive F-groups I, H, define

$$\mathcal{D}(I, H; F) := \ker[H^1(F, I) \to H^1(F, H)].$$

For the remaining of the section, we make the following assumption:

- Our choice of $\delta \in G(F)$ (resp. of $\gamma \in H(F)$) is such that $C^\tau_{\delta\delta^{-\theta},G^\sigma}$ (resp. $C_{\gamma\gamma^{-\sigma},G^\sigma}$) is connected.

In the following section, we will prove that this is always the case provided δ (resp. γ) is relatively τ-regular semisimple (resp. relatively regular semisimple).

We have the following result:

Lemma 3.1.4. *Let $\delta \in G(F)$. The (pointed) set $\mathcal{D}(C^\tau_{\delta\delta^{-\theta},G^\sigma}, G^\sigma; F)$ parametrises the τ-relative classes inside the stable τ-relative class of δ.*

Proof. Let $g \in G^\sigma(\overline{F})$ be such that $g\delta\delta^{-\theta}g^{-\tau} \in G(F)$. Then, for all $\varepsilon \in \Gamma$, we have

$$\begin{aligned}(g^{-1}\varepsilon(g))\delta\delta^{-\theta}(g^{-1}\varepsilon(g))^{-\tau} &= g^{-1}\varepsilon(g)\delta\delta^{-\theta}\varepsilon(g^{-\tau})g^{\tau}\\ &= g^{-1}\varepsilon(g\delta\delta^{-\theta}g^{-\tau})g^{\tau}\\ &= g^{-1}(g\delta\delta^{-\theta}g^{-\tau})g^{\tau}\\ &= \delta\delta^{-\theta},\end{aligned}$$

and therefore, we see that $g^{-1}\varepsilon(g) \in C^{\tau}_{\delta\delta^{-\theta},G^\sigma}(\overline{F})$. Hence, we can define a cocyle $\{\varepsilon \mapsto g^{-1}\varepsilon(g)\}$, which clearly lies in $\mathcal{D}(C^{\tau}_{\delta\delta^{-\theta},G^\sigma}, G^\sigma; F)$. Conversely, given $g \in G^\sigma(\overline{F})$ such that $g^{-1}\varepsilon(g) \in C^{\tau}_{\delta\delta^{-\theta},G^\sigma}(\overline{F})$ for all $\varepsilon \in \Gamma$ (we can assume the cocyle has this form, since it is trivial in $H^1(F, G^\sigma)$), then

$$(g^{-1}\varepsilon(g))\delta\delta^{-\theta}(g^{-1}\varepsilon(g))^{-\tau} = \delta\delta^{-\theta} \Longrightarrow \varepsilon(g\delta\delta^{-\theta}g^{-\tau}) = g\delta\delta^{-\theta}g^{-\tau},$$

for all $\varepsilon \in \Gamma$, and so $g^{-1}\gamma g^{\sigma} \in G^\sigma(F)$. Hence, we have a surjection

$$\{\text{Stable } \tau\text{-relative class of } \delta\} \twoheadrightarrow \mathcal{D}(C^{\tau}_{\delta\delta^{-\theta},G^\sigma}, G^\sigma; F).$$

Now, given $g, \tilde{g} \in G^\sigma(\overline{F})$ such that $g\delta\delta^{-\theta}g^{-\tau}, \tilde{g}\delta\delta^{-\theta}\tilde{g}^{-\tau} \in G(F)$, suppose there exists $h \in G^\sigma(F)$ such that

$$g\delta\delta^{-\theta}g^{-\tau} = h(\tilde{g}\delta\delta^{-\theta}\tilde{g}^{-\tau})h^{-\tau}.$$

From this, we immediately get that $g^{-1}h\tilde{g} \in C^{\tau}_{\delta\delta^{-\theta},G^\sigma}(\overline{F})$. Then, for all $\varepsilon \in \Gamma$, we have

$$(g^{-1}h\tilde{g})^{-1}(g^{-1}\varepsilon(g))\varepsilon(g^{-1}h\tilde{g}) = \tilde{g}^{-1}\varepsilon(\tilde{g}),$$

and so $\{\varepsilon \mapsto g^{-1}\varepsilon(g)\}, \{\varepsilon \mapsto \tilde{g}^{-1}\varepsilon(\tilde{g})\}$ are cohomologous. Conversely, if there exists $c \in C^{\tau}_{\delta\delta^{-\theta},G^{\sigma}}(\overline{F})$ such that $c^{-1}(g^{-1}\varepsilon(g))\varepsilon(c) = \tilde{g}^{-1}\varepsilon(\tilde{g})$ for all $\varepsilon \in \Gamma$, then $gc\tilde{g}^{-1} \in G^{\sigma}(F)$ and we see that

$$\begin{aligned}(gc\tilde{g}^{-1})(\tilde{g}\delta\delta^{-\theta}\tilde{g}^{-\tau})(gc\tilde{g}^{-1})^{-\tau} &= gc\tilde{g}^{-1}(\tilde{g}\delta\delta^{-\theta}\tilde{g}^{-\tau})\tilde{g}^{\tau}c^{-\tau}g^{-\tau} \\ &= g(c\delta\delta^{-\theta}c^{-\tau})g^{-\tau} \\ &= g\delta\delta^{-\theta}g^{-\tau},\end{aligned}$$

and so $g\delta\delta^{-\theta}g^{-\tau}$ and $\tilde{g}\delta\delta^{-\theta}\tilde{g}^{-\tau}$ are in the same τ-relative class. □

If we let τ be trivial in the above computations, we also deduce that $\mathcal{D}(C_{\gamma\gamma^{-\sigma},H^{\sigma}}, H^{\sigma}; F)$ parametrizes the relative classes inside the stable relative class of γ. By a well known Galois cohomological result, the sets $\mathcal{D}(C_{\gamma\gamma^{-\sigma},H^{\sigma}}, H^{\sigma}; F)$ and $\mathcal{D}(C^{\tau}_{\delta\delta^{-\theta},G^{\sigma}}, G^{\sigma}; F)$ are finite whenever F is a local field (Ser97, §III.4, Théorème 4).

3.2 Definition of a norm map

We are now ready to match the relative classes and τ-classes.

Definition 3.2.1. We say that γ is a *norm* of δ if there exists an element $g \in G^{\sigma}(\overline{F})$ such that

$$g\delta\delta^{-\theta}(\delta\delta^{-\theta})^{\tau}g^{-1} = \gamma\gamma^{-\sigma}.$$

We note that

$$\delta\delta^{-\theta}(\delta\delta^{-\theta})^{\tau} = \delta\delta^{-\theta}(\delta\delta^{-\theta})^{-\sigma}.$$

Also, if γ is a norm of δ, we have an inner twist $C^{\tau}_{\delta\delta^{-\theta},G^{\sigma}} \cong C_{\gamma\gamma^{-\theta},H^{\sigma}}$.

We first recall the following lemma, which is a variation on the proof of (GW, Lemma 3.6) to the situation under consideration.

Lemma 3.2.2. *(Get, Lemma 3.12) Let $\alpha \in Q(F)$ be regular semisimple. The torus $T = C_{\alpha,H}$ is maximal σ-stable, and we let T_σ be its maximal σ-split subtorus (Ric82, §1). The map*

$$T(F) \to T_\sigma(F)$$
$$t \mapsto tt^{-\sigma}$$

is surjective and α is in the image.

Proof. Since $\alpha \in H(F)$ is σ-split, we can easily check that $C_{\alpha,H}$ is σ-stable, and the fact that it is a maximal torus follows from α being a regular semisimple element. Also, we note that T_σ is the scheme-theoretic image of the moment map

$$B_\sigma : T(R) \to T(R)$$
$$t \mapsto tt^{-\sigma};$$

indeed, the restriction $B_\sigma|_{T_\sigma}$ is surjective, since it coincides with $t \mapsto t^2$, and the image of B_σ is contained in T_σ.

Let $U = \mathrm{Res}_{M/F}T$ and let $A = \{t \in U(F) : t^{-\tau} = t\}$. Since τ and σ commute, we see that A is σ-stable. Moreover, we have a short exact sequence:

$$1 \to U(F) \to T(F) \to 1$$
$$t \mapsto tt^{\tau},$$

where the surjectivity of the last map is given by (Rog90, Proposition 3.11.1). We can thus form an exact hexagon:

$$\begin{array}{ccccc} & H^0(\langle\sigma\rangle, A) & \xrightarrow{\ a\ } & H^0(\langle\sigma\rangle, U(F)) & \\ {}^{b}\nearrow & & & & \searrow \\ H^1(\langle\sigma\rangle, T(F)) & & & & H^0(\langle\sigma\rangle, T(F)) \\ {}_{c}\nwarrow & & & & \swarrow \\ & H^1(\langle\sigma\rangle, U(F)) & \longleftarrow & H^1(\langle\sigma\rangle, A) & \end{array}$$

(Ser68, §VIII.4). Since the map a is injective, the map b is the trivial map. Using the notation of (Ser68, §VIII.1 and §VIII.4), we have isomorphisms

$$N_{\langle\sigma\rangle}U(F)/I_{\langle\sigma\rangle}U(F) \cong \widehat{H}^{-1}(\langle\sigma\rangle, U(F)) \cong H^1(\langle\sigma\rangle, U(F)).$$

Let $U_\sigma \leq U$ be the maximal σ-split subtorus. We note that

$$N_{\langle\sigma\rangle}U(F)/I_{\langle\sigma\rangle}U(F) = 1$$

if and only if the map

$$\begin{aligned} U(F) &\to U(F) \\ t &\mapsto tt^{-\sigma} \end{aligned} \tag{3.2.0.1}$$

is surjective. If we let $U^\sigma \leq U$ be the subtorus fixed by σ, the surjectivity follows from (Ser68, §X.1, Exercise 2), since the fibers of the map 3.2.0.1 are torsors over U^σ. Therefore, we have

$$H^1(\langle\sigma\rangle, U(F)) \cong N_{\langle\sigma\rangle}U(F)/I_{\langle\sigma\rangle}U(F) = 1.$$

In turn, this implies that the map c in the hexagon is also the trivial map. It now follows that

$$1 = H^1(\langle\sigma\rangle, T(F)) \cong \widehat{H}^{-1}(\langle\sigma\rangle, T(F)) \cong N_{\langle\sigma\rangle}T(F)/I_{\langle\sigma\rangle}T(F),$$

and we can therefore conclude that the moment map $B_\sigma : T \to T$ is surjective on F-points.

Finally, we want to show that α is in the image of this map. We know that α is contained in a maximal σ-split torus T'_σ of $H_{\overline{F}}$ (Ric82, Theorem 7.5), which is itself contained in a maximal σ-stable torus T' (Hel91, Proposition 1.4). Moreover, by (Ric82, Theorem 7.5), T'_σ is the unique maximal σ-split torus of T'. Since α is a regular semisimple element, it is contained in a unique maximal torus, and we therefore deduce that $T' = T_{\overline{F}}$ and $T'_\sigma = T_{\sigma\overline{F}}$. It now follows that $\alpha \in T_\sigma(F)$.

□

We now prove the following proposition.

Proposition 3.2.3. *Let $\gamma \in H(F)$ be relatively regular semisimple. Then there exists $\delta \in G(F)$ such that $\delta\delta^{-\theta}(\delta\delta^{-\theta})^\tau = \gamma\gamma^{-\sigma}$.*

Proof. By 3.2.2, there exists $\beta \in C_{\gamma\gamma^{-\sigma},H}(F)$ such that $\gamma\gamma^{-\sigma} = \beta\beta^{-\sigma}$. Let $T = \mathrm{Res}_{M/F}(C_{\gamma\gamma^{-\sigma},H}) \leq G$. By (Rog90, Proposition 3.11.1), there exists $\delta \in T(F)$ such that $\beta = \delta\delta^\tau$. Since T is commutative, we have

$$\delta\delta^{-\theta}(\delta\delta^{-\theta})^\tau = \delta\delta^\tau(\delta\delta^\tau)^{-\sigma} = \beta\beta^{-\sigma} = \gamma\gamma^{-\sigma},$$

and the result follows. □

It thus remains to prove that every $\delta \in G(F)$ admits a norm (we will actually prove a weaker result). This will occupy the remaining sections. Write

$$N(\delta) := \delta\delta^{-\theta}(\delta\delta^{-\theta})^{\tau}.$$

Since the group $C_{N(\delta),G^{\sigma}}$ is exactly the unit group of an algebra over F (namely, the centralizer in the algebra $\mathrm{Mat}_{2n}(\overline{F}) \times \mathrm{Mat}_{2n}(\overline{F})$), the set $\mathcal{D}(C_{N(\delta),G^{\sigma}}, G^{\sigma}; F)$ is always trivial (Ser68, §X.1, Exercise 2). We will therefore follow the following strategy: first, we will prove that a $G^{\sigma}(F)$-conjugate of $N(\delta)$ lies in $Q(F)$, and then we will show that this conjugate has the desired form.

3.3 Preliminary Computations

We now assume the extension M/F is unramified. Local class field theory (Ser68, Chapitre V, Proposition 3) thus tells us that the norm map restricted to the units is *surjective*, that is,

$$\mathrm{Nm}_{M/F}(\mathcal{O}_M^{\times}) = \mathcal{O}_F^{\times}.$$

We therefore have the following useful lemma.

Lemma 3.3.1. *An element $x \in F^{\times}$ is a norm if and only if* $\mathrm{val}(x)$ *is even. In particular, there exists $u \in M^{\times}$ such that*

$$x = \pi_F^{\epsilon}\mathrm{Nm}_{M/F}(u),$$

where π_F is a uniformiser for F and $\epsilon \in \{0, 1\}$ is the residue of $val(x)$ modulo 2.

Proof. Consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_M^\times & \longrightarrow & M^\times & \xrightarrow{val} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Nm}} & & \downarrow{\scriptstyle \mathrm{Nm}} & & \downarrow{\scriptstyle \times 2} & & \\
0 & \longrightarrow & \mathcal{O}_F^\times & \longrightarrow & F^\times & \xrightarrow{val} & \mathbb{Z} & \longrightarrow & 0,
\end{array}$$

where the first two vertical maps are given by the norm map. Since M/F is unramified, we know that the first vertical map is surjective, and so the Snake Lemma implies that

$$F^\times/\mathrm{Nm}_{M/F}(M^\times) \cong \mathbb{Z}/2\mathbb{Z}.$$

Now, recall that $F^\times \cong \mathcal{O}_F^\times \times \mathbb{Z}$, where $x \mapsto (x\pi_F^{-val(x)}, val(x))$. But since $\mathcal{O}_F^\times \subset \mathrm{Nm}_{M/F}(M^\times)$, we see that $\mathrm{Nm}_{M/F}(M^\times) \cong \mathcal{O}_F^\times \times 2\mathbb{Z}$, and the result follows. □

We define the following map

$$\tilde{B}_\sigma : G \to G,$$

which on points sends an element g to the product $gg^{-\sigma}$; we also let $\tilde{Q}$ be the scheme-theoretic image of this map. The group G^σ acts on $\tilde{Q}$ by conjugation. We thus wish to answer the following question:

Question 3.3.2. *Which semisimple orbits in $\tilde{Q}/G^\sigma$ have a representative in H?*

Following (JR96, Proposition 4.1), we have the following characterisation of semisimple orbits:

Proposition 3.3.3. *Semisimple orbits are parametrised by triples $(\nu, \{A\}, \beta)$, where $0 \leq \nu \leq 2n$ is an integer, $\{A\}$ is a semisimple conjugacy class in $\mathrm{Mat}_\nu(M)$ without* *the eigenvalues* ± 1*, and* β *is an integer with* $0 \le \beta \le 2n-\nu$*. Moreover, such a triple corresponds to the following canonical representative:*

$$\begin{pmatrix} A & 0 & I_\nu & 0 \\ 0 & \eta & 0 & 0 \\ A^2 - I_\nu & 0 & A & 0 \\ 0 & 0 & 0 & \eta \end{pmatrix};$$

the matrix η *is of the form*

$$\begin{pmatrix} I_\alpha & \\ & -I_\beta \end{pmatrix},$$

where $\alpha + \beta = 2n - \nu$*.*

We can draw a few consequences from this result. Let $t(\nu, \{A\}, \beta)$ denote the canonical form associated to the triple $(\nu, \{A\}, \beta)$. Since $t(\nu, \{A\}, \beta)$ is σ-split, if λ is an eigenvalue, so is λ^{-1}, and we have a two-to-one surjective map (JR96, Lemma 4.3)

$$\left\{ \begin{array}{c} \textbf{Eigenvalues of} \\ t(\nu, \{A\}, \beta) \\ \textbf{(with multiplicity)} \end{array} \right\} \to \left\{ \begin{array}{c} \textbf{Eigenvalues of} \\ \begin{pmatrix} A & 0 \\ 0 & \eta \end{pmatrix} \\ \textbf{(with multiplicity)} \end{array} \right\}$$

$$\lambda \mapsto \left(\frac{\lambda + \lambda^{-1}}{2} \right).$$

If we assume $t(\nu, \{A\}, \beta)$ is regular semisimple, we claim that ± 1 is not an eigenvalue. Indeed, as noted in (JR96, Lemma 4.3), if T is an eigenvector with eigenvalue ± 1, then so is $I_{2n,2n}T$, which implies that $I_{2n,2n}T = \alpha T$ for some scalar α. We can write $T = v + w$, where v (resp. w) is in and eigenvector of $I_{2n,2n}$ with eigenvalue 1 (resp. -1). We thus get

$$I_{2n,2n}T = \alpha T \Rightarrow v - w = \alpha v + \alpha w \Rightarrow (\alpha - 1)v + (\alpha + 1)w = 0.$$

But since v and w are linearly independent, we get that $\alpha = 1$ and $\alpha = -1$, which is absurd; the claim follows. Therefore, we conclude that $t(\nu, \{A\}, \beta)$ is regular semisimple if and only if $\nu = 2n, \beta = 0$ and $\{A\}$ is itself regular semisimple.

A simple computation then shows that

$$C_{t(2n,\{A\},0),\mathrm{GL}_{2n}\times\mathrm{GL}_{2n}} \cong C_{A,\mathrm{GL}_{2n}}.$$

Since GL_{2n} has simply-connected derived subgroup, both groups are connected (Kot82, §3). By passing to the algebraic closure, we can see that this has the following consequence (cf. the discussion preceding Lemma 3.1.4):

Proposition 3.3.4. *Suppose $\delta \in G(F)$ (resp. $\gamma \in H(F)$) is relatively τ-regular semisimple (resp. relatively regular semisimple). Then $C^{\tau}_{\delta\delta^{-\theta},G^{\sigma}}$ (resp. $C_{\gamma\gamma^{-\sigma},G^{\sigma}}$) is connected.*

We now wish to answer the question we raised above. We shall henceforth assume that $n = 1$ (and therefore $G(F) = \mathrm{GL}_4(M)$). Hence, we have six basic cases to consider, depending on the values of ν and β:

1. $\nu = 0$ and $\beta = 0$;
2. $\nu = 0$ and $\beta = 1$;
3. $\nu = 0$ and $\beta = 2$;

4. $\nu = 1$ and $\beta = 0$;

5. $\nu = 1$ and $\beta = 1$;

6. $\nu = 2$ and $\beta = 0$.

Note that the last case corresponds to the regular semisimple case. The first three cases are trivial: the canonical forms are diagonal matrices with ± 1 on the diagonal, and these are easily seen to be already in H.

Before treating the remaining cases, we make the following observation. We want to know under what conditions a $\mathrm{GL}_2(M) \times \mathrm{GL}_2(M)$-conjugate of the matrix

$$g := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \tilde{Q}(F)$$

lies in $H(F) = \mathrm{GL}_4(M)^\tau$. First, we note that since g (and all its conjugates) is σ-split, then it lies in $H(F)$ if and only if it θ-split. This condition readily implies that in order for a conjugate of g to lie in $H(F)$, we need the characteristic polynomial of A to have coefficients in F (and not merely in M); we therefore assume that this is always the case. Now, write

$$g_1 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, g_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(M).$$

We want g_1, g_2 to be such that

$$\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} g_1^{-1} & \\ & g_2^{-1} \end{pmatrix} = \begin{pmatrix} g_1 A g_1^{-1} & g_1 B g_2^{-1} \\ g_2 C g_1^{-1} & g_2 D g_2^{-1} \end{pmatrix} \in H(F).$$

Hence, this translates to

$$
\begin{aligned}
A^{-\tau} g_1^{-\tau} g_1 &= g_1^{-\tau} g_1 A, \\
D^{-\tau} g_2^{-\tau} g_2 &= g_2^{-\tau} g_2 D, \\
g_2^{-\tau} g_2 C &= -B^{-\tau} g_1^{-\tau} g_1.
\end{aligned} \tag{3.3.0.2}
$$

These equations will underlie all the remaining computations.

3.3.1 The cases where $\nu = 1$

The next two cases are similar, and can be treated simultaneously. We note that the canonical form given by Jacquet-Rallis is

$$
g = \begin{pmatrix} x & & 1 & \\ & \pm 1 & & \\ x^2 - 1 & & x & \\ & & & \pm 1 \end{pmatrix},
$$

where $x \in M$ is different from ± 1. Note that since the determinant of the upper-left block matrix lies in F, this implies that $x \in F$. If we rewrite Equation 3.3.0.2 in terms of the canonical form g, we get (after simplification)

$$
\begin{cases}
\overline{\alpha}\beta = \overline{\gamma}\delta \\
\overline{a}b = \overline{c}d \\
(x^2 - 1)(\mathrm{Nm}_{M/F}(a) - \mathrm{Nm}_{M/F}(c)) = \mathrm{Nm}_{M/F}(\gamma) - \mathrm{Nm}_{M/F}(\alpha).
\end{cases}
$$

We now have the following result:

Proposition 3.3.5. *The above matrix has a G^σ-conjugate in H.*

Proof. Suppose $x^2 - 1 = \mathrm{Nm}_{M/F}(z)$, for some $z \in M$. Then, conjugating g by

$$\begin{pmatrix} x & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{pmatrix} \in G^\sigma(F)$$

gives a unitary matrix. On the other hand, if $x^2 - 1$ is not a norm, we can write $x^2 - 1 = \pi_F \mathrm{Nm}_{M/F}(u)$ (by Lemma 3.3.1). In this case, we can take

$$g_1 = \begin{pmatrix} 1 & \beta \\ \overline{\beta} & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$

where $\mathrm{Nm}_{M/F}(\beta) = 1 + \pi_F$; such β exists since $1 + \pi_F \in \mathcal{O}_F^\times = \mathrm{Nm}_{M/F}(\mathcal{O}_M^\times)$. □

3.3.2 The cases where $\nu = 2$

We now want to know under what conditions a $\mathrm{GL}_2(M) \times \mathrm{GL}_2(M)$-conjugate of the matrix

$$g := \begin{pmatrix} A & I \\ A^2 - I & A \end{pmatrix}$$

lies in $H(F) = \mathrm{GL}_4(M)^\tau$, where $A \in \mathrm{GL}_2(M)$ is a semisimple matrix without eigenvalues ± 1. Recall that we have assumed the characteristic polynomial of A has coefficients in F. Hence, we have three different possibilities for the eigenvalues: either they are in F, or they generate a quadratic extension of F which is equal to M, or distinct from M. We only treat the first two cases.

1. **The eigenvalues lie in F**

Let $x, y \in F$ be the eigenvalues of A, and so we can assume that

$$A = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

In this case, Equation 3.3.0.2 now takes the form (after simplification)

$$\begin{cases} x(\overline{\alpha}\beta - \overline{\gamma}\delta) = y(\overline{\alpha}\beta - \overline{\gamma}\delta) \\ x(\overline{a}b - \overline{c}d) = y(\overline{a}b - \overline{c}d) \\ (x^2 - 1)(\mathrm{Nm}_{M/F}(a) - \mathrm{Nm}_{M/F}(c)) = \mathrm{Nm}_{M/F}(\gamma) - \mathrm{Nm}_{M/F}(\alpha) \\ (x^2 - 1)(c\overline{d} - a\overline{b}) = \alpha\overline{\beta} - \gamma\overline{\delta} \\ (y^2 - 1)(\mathrm{Nm}_{M/F}(d) - \mathrm{Nm}_{M/F}(b)) = \mathrm{Nm}_{M/F}(\beta) - \mathrm{Nm}_{M/F}(\delta) \\ (y^2 - 1)(\overline{a}b - \overline{c}d) = \overline{\gamma}\delta - \overline{\alpha}\beta. \end{cases}$$

Suppose there exists $z, w \in M$ such that $\mathrm{Nm}_{M/F}(z) = x^2 - 1$ and $\mathrm{Nm}_{M/F}(w) = y^2 - 1$. If we set

$$g_1 = \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}, g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

a computation similar to the one above shows that g_1, g_2 satisfy the equations above, and so in that case g has a conjugate in $H(F)$. More generally, we have the following

Theorem 3.3.6. *Assume that* $\det(A^2 - I)$ *is a norm. Then the matrix g has a conjugate in $H(F)$.*

Proof. The discussion preceding this theorem gives a proof in the case where both $x^2 - 1$ and $y^2 - 1$ are norms. If neither is a norm, write

$$x^2 - 1 = \pi_F \mathrm{Nm}_{M/F}(u), \qquad y^2 - 1 = \pi_F \mathrm{Nm}_{M/F}(v),$$

where $u, v \in M^\times$. Then, we can take

$$g_1 = \begin{pmatrix} 1 & \beta \\ \overline{\beta} & 1 \end{pmatrix}, g_2 = \begin{pmatrix} u^{-1} & 0 \\ 0 & v^{-1} \end{pmatrix},$$

where again β is such that $\mathrm{Nm}_{M/F}(\beta) = 1 + \pi_F$. Finally, the condition on the determinant ensures that these are all the cases we have to consider. □

2. **The eigenvalues (properly) lie in M**

Let $x \in M$ be one of these eigenvalues (the other one is $\overline{x}$). Hence, we can assume that

$$A = \begin{pmatrix} x & 0 \\ 0 & \overline{x} \end{pmatrix}.$$

In this case, the above matrix equations 3.3.0.2 translate to

$$\begin{cases} \mathrm{Nm}_{M/F}(\alpha) = \mathrm{Nm}_{M/F}(\gamma), \\ \mathrm{Nm}_{M/F}(\delta) = \mathrm{Nm}_{M/F}(\beta), \\ \mathrm{Nm}_{M/F}(a) = \mathrm{Nm}_{M/F}(c), \\ \mathrm{Nm}_{M/F}(d) = \mathrm{Nm}_{M/F}(b), \\ (x^2 - 1)\left(\gamma\overline{\delta} - \alpha\overline{\beta}\right) = \left(a\overline{b} - c\overline{d}\right). \end{cases}$$

Again, we have a positive result.

Theorem 3.3.7. *The matrix g has a conjugate in $H(F)$.*

Proof. If we take

$$g_1 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 1-x^2 & -1 \\ x^2-1 & -1 \end{pmatrix} \in \mathrm{GL}_2(M),$$

then a simple computation shows that g_1, g_2 indeed satisfy Equation 3.3.0.2. □

3.4 Norm computation

We are now ready to go back to the case where $g = N(\delta)$, with $\delta \in G(F)$ relatively τ-regular semisimple. First, we note that for all $\delta \in G(F)$, we have

$$\delta\delta^{-\theta} = \begin{pmatrix} A & B \\ -B^{-\tau} & D \end{pmatrix},$$

where A, D are fixed by $-\tau$.

Depending on the rank of B, we have three different cases. However, since the upper-right corner of $\delta\delta^{-\theta}(\delta\delta^{-\theta})^{\tau}$ has to be full-rank (cf. Proposition 3.3.3), this implies that the only case possible is

$$\delta\delta^{-\theta} = \begin{pmatrix} A & I \\ -I & D \end{pmatrix}.$$

In this case, we can easily compute $\delta\delta^{-\theta}(\delta\delta^{-\theta})^{\tau}$. First, write

$$(\delta\delta^{-\theta})^{-1} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}.$$

Therefore, we have

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & I \\ -I & D \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$
$$= \begin{pmatrix} AX + Z & AY + W \\ DZ - X & DW - Y \end{pmatrix}.$$

Hence, we see that $X = DZ$, and so

$$AX + Z = I \Rightarrow A(DZ) + Z = I \Rightarrow (AD + I)Z = I.$$

Similarly, we have $W = -AY$, and so we get

$$DW - Y = I \Rightarrow D(-AY) - Y = I \Rightarrow -(DA + I)Y = I.$$

The conclusion is that

$$(\delta\delta^{-\theta})^{-1} = \begin{pmatrix} D(AD + I)^{-1} & -(DA + I)^{-1} \\ (AD + I)^{-1} & A(DA + I)^{-1} \end{pmatrix},$$

and so

$$\delta\delta^{-\theta}(\delta\delta^{-\theta})^{\tau} = \begin{pmatrix} (AD - I)(AD + I)^{-1} & 2A(DA + I)^{-1} \\ -2D(AD + I)^{-1} & (DA - I)(DA + I)^{-1} \end{pmatrix}.$$

Remark. It is straightforward to show that $\det(\delta\delta^{-\theta}) = \det(AD + I) = \det(DA + I)$, and so the matrices being inverted above are indeed invertible.

3.4.1 Cayley transform

The matrix appearing in the upper-left corner of $gg^{-\sigma}$ is the *Cayley transform* of AD. In this subsection, we will study the properties of the Cayley transform.

Definition 3.4.1. Let P be an $n \times n$-matrix with coefficients in M. Assume that -1 is not an eigenvalue of P. Then $P - I$ is invertible, and we define the *Cayley transform* of P as

$$c(P) = (P - I)(P + I)^{-1}.$$

Note that we also have $c(P) = (P + I)^{-1}(P - I)$.

If P is semisimple, so is $c(P)$, and their eigenvalues are related in a simple way.

Lemma 3.4.2. *If $\{\lambda_1, \dots, \lambda_n\}$ are the eigenvalues of P, then $\{\mu_1, \dots, \mu_n\}$ are the eigenvalues of $c(P)$, where*

$$\mu_i = \frac{\lambda_i - 1}{\lambda_i + 1}.$$

Proof. If $v \in M^n$ is such that $Pv = \lambda v$, then

$$c(P)v = (P - I)(P + I)^{-1}v = (P - I)((\lambda + 1)^{-1}v) = (\lambda - 1)(\lambda + 1)^{-1}v.$$

Conversely, if v is such that $c(P)v = \mu v$, we have

$$\begin{aligned}(P - I)(P + I)^{-1}v = \mu v &\Rightarrow (P + I)^{-1}(P - I)v = \mu v \\ &\Rightarrow (P - I)v = \mu(P + I)v \\ &\Rightarrow Pv - v = \mu Pv + \mu v \\ &\Rightarrow Pv = (1 + \mu)(1 - \mu)^{-1}v.\end{aligned}$$

□

Corollary 3.4.3. *The eigenvalues of P generate the same field extension as the eigenvalues of $c(P)$.*

3.4.2 Existence of norms

We are now ready to prove the main theorem of this section:

Theorem 3.4.4. *Let $\delta \in G(\mathcal{O}_F)$ be such that*

$$\delta\delta^{-\theta} = \begin{pmatrix} A & I \\ -I & D \end{pmatrix},$$

and assume the eigenvalues of AD lie in M. Then δ admits a norm.

Proof. In view of the results of the preceding section, we first need to show that $\delta\delta^{-\theta}(\delta\delta^{-\theta})^{\tau}$ satisfies the various hypotheses.

(1) The characteristic polynomial of $c(AD)$ has coefficients in F

Let $\{\lambda, \mu\}$ be the eigenvalues of AD. Then the previous proposition shows that the eigenvalues of $c(AD)$ are $\left\{\frac{\lambda-1}{\lambda+1}, \frac{\mu-1}{\mu+1}\right\}$. Hence, we have that

$$\mathrm{Tr}(c(AD)) = \frac{\lambda-1}{\lambda+1} + \frac{\mu-1}{\mu+1} = \frac{2\mathrm{Tr}(AD)-2}{\det(AD)+\mathrm{Tr}(AD)+1}$$

and

$$\det(c(AD)) = \left(\frac{\lambda-1}{\lambda+1}\right)\cdot\left(\frac{\mu-1}{\mu+1}\right) = \frac{\det(AD)-\mathrm{Tr}(AD)+1}{\det(AD)+\mathrm{Tr}(AD)+1}.$$

On the other hand, we have

$$\mathrm{Tr}(AD) = \mathrm{Tr}(A^{-\tau}D^{-\tau}) = \mathrm{Tr}\left(\overline{(DA)}^{t}\right) = \overline{\mathrm{Tr}(AD)}$$

and similarly

$$\det(AD) = \det(A^{-\tau})\det(D^{-\tau}) = \overline{\det(A)} \cdot \overline{\det(D)} = \overline{\det(AD)};$$

the claim thus follows.

(2) The determinant of $(c(AD)^2 - I)$ is a norm

We have

$$\begin{aligned}\det(c(AD)^2 - I) &= \left(\left(\frac{\lambda-1}{\lambda+1}\right)^2 - 1\right)\left(\left(\frac{\mu-1}{\mu+1}\right)^2 - 1\right) \\ &= \left(\left(\frac{\lambda-1}{\lambda+1}\right) - 1\right)\left(\left(\frac{\lambda-1}{\lambda+1}\right) + 1\right)\left(\left(\frac{\mu-1}{\mu+1}\right) - 1\right)\left(\left(\frac{\mu-1}{\mu+1}\right) + 1\right) \\ &= \left(\frac{-2}{\lambda+1}\right)\left(\frac{2\lambda}{\lambda+1}\right)\left(\frac{-2}{\mu+1}\right)\left(\frac{2\mu}{\mu+1}\right).\end{aligned}$$

Therefore, we have

$$\mathrm{val}(\det(c(AD)^2 - I)) = (\mathrm{val}(\lambda) + \mathrm{val}(\mu)) - 2\left(\mathrm{val}(\lambda+1) + \mathrm{val}(\mu+1)\right),$$

which is even, hence $\det(c(AD)^2 - I)$ is a norm (Lemma 3.3.1).

Finally, by 3.2.2, there exists $\gamma \in H(F)$ such that

$$\begin{pmatrix} h & \\ & h \end{pmatrix} N(\delta) \begin{pmatrix} h^{-1} & \\ & h^{-1} \end{pmatrix} = \gamma\gamma^{-\sigma}.$$

This concludes the proof.

□

CHAPTER 4
A Relative Fundamental Lemma

In the previous chapter, we presented the various objects that are at the heart of this thesis. We then proved a matching statement between the twisted relative classes in our general linear group and the relative classes in our unitary group. The next step would be to match the orbital integrals that appear on the geometric side of our trace formulas. We define this notion of matching (which is the so-called *relative fundamental lemma*) and we prove it in the "depth zero" case. We keep the same notation as in the previous chapter, except for the first section. In particular, M/F is a quadratic extension of non-archimedean local fields (we make the further assumption that the residual characteristic is different from two), $G = \mathrm{Res}_{M/F}(\mathrm{GL}_{4n})$, and H is a unitary group over F in $4n$ variables.

4.1 Integration over locally compact groups

As we alluded to earlier, if F is a locally compact field and G is an algebraic group over F, the group $G(F)$ inherits a natural topology coming from that of F. As such, $G(F)$ is a locally compact topological group, and so it admits a (left) *Haar measure* which is unique up to multiplication by a positive real number. If it is also a right Haar measure, we say $G(F)$ (or simply G) is *unimodular*.

Lemma 4.1.1. *If the group G is a connected reductive group, then $G(F)$ is unimodular.*

Proof. Recall that the modular character $\Delta : G(F) \to \mathbb{R}^+$ is a continuous group homomorphism and that $G(F)$ is unimodular if and only if Δ is trivial. By definition, Δ is trivial on the center Z of $G(F)$ and on its derived subgroup $G(F)'$ (since $\mathbb{R}^+$ is abelian). By Lemma 19.5 and Theorem 27.5(d) of (Hum75), $ZG(F)'$ has finite index in $G(F)$. But $\mathbb{R}^+$ has no non-trivial finite subgroup, and thus the result follows. □

In the later sections of this chapter, we will need to integrate over homogeneous spaces (i.e. the quotient of a topological group by a closed subgroup). We do have Haar measures on our groups, but it is not *a priori* clear that we can get an invariant measure on our homogeneous space. The precise case when this happens is described in the following lemma:

Lemma 4.1.2. *(PR94, Theorem 3.17) Let G be a unimodular locally compact Hausdorff topological group, and let H be a closed subgroup. Then there exists a right G-invariant Radon measure on $H\backslash G$ if and only if H is unimodular.* □

For the rest of this section, we will assume the hypotheses of the previous lemma are satisfied, and that H is unimodular. Moreover, we will also assume that G (and hence H) is totally disconnected; we thus have a neighbourhood basis of the identity consisting of open compact subgroups.

Choose Haar measures dg and dh on G and H, respectively. Then we can choose an invariant measure $\frac{dg}{dh}$ on $H\backslash G$ such that

$$\int_G f(g)dg = \int_{H\backslash G}\int_H f(hg)dh\frac{dg}{dh} \tag{4.1.0.1}$$

for all $f \in C_c^\infty(G)$. This formula characterizes the invariant measure, since the map

$$C_c^\infty(G) \to C_c^\infty(H\backslash G)$$
$$f \mapsto f^\#,$$

where $f^\#(gH) = \int_H f(hg)dh$, is surjective. Using this characterization, we can compute the volume of all compact open subsets of $H\backslash G$: such a set is a disjoint union of subsets of the form $H\backslash HgK$, where K is a compact open subgroup of G. By taking f to be the characteristic function of gK in Equation 4.1.0.1, we see that

$$\mathrm{vol}(H\backslash HgK) = \frac{\mathrm{vol}(K)}{\mathrm{vol}(H \cap gKg^{-1})}. \tag{4.1.0.2}$$

This fact will be used in Lemma 4.3.2 below to simplify our relative orbital integrals.

4.2 Definitions of local orbital integrals

Recall from Chapter 2 that the geometric side of the trace formula involves orbital integrals, which are *local* objects. Hence, in this section, we define *local* orbital integrals, which are the objects appearing in the relative fundamental lemma. We first recall the following result.

Lemma 4.2.1. *(Hah09, Theorem 2.5) Let $\delta \in G(F)$, $\gamma \in H(F)$. If δ is relatively τ-semisimple (resp. if γ is relatively semisimple), then $C^\tau_{\delta\delta^{-\theta},G^\sigma}$ (resp. $C_{\gamma\gamma^{-\sigma},H^\sigma}$) is reductive.* □

Let $\phi \in C_c^\infty(H(F))$ and let $\gamma \in H(F)$ be relatively semisimple, and assume that $C_{\gamma\gamma^{-\sigma},H^\sigma}$ is connected. The *local relative orbital integral* for γ is given by

$$O_\gamma(\phi) := \int_{H_\gamma(F)\backslash H^\sigma(F)^2} \phi(h_1^{-1}\gamma h_2)\frac{dh_1 dh_2}{dt_\gamma},$$

where dh_i and dt_γ are Haar measures on $H^\sigma(F)$ and $H_\gamma(F)$, respectively. Similarly, if $f \in C_c^\infty(G(F))$ and $\delta \in G(F)$ is relatively τ-semisimple (again, we assume $C^\tau_{\delta\delta^{-\theta},G^\sigma}$ is connected), we define the *local relative twisted orbital integral* for δ by

$$TO_\delta(f) := \int_{G_\delta(F)\backslash G^\sigma \times G^\theta(F)} f(g_1^{-1}\delta g_2)\frac{dg_1 dg_2}{dt_\delta},$$

where dg_1, dg_2 and dt_δ are Haar measures on $G^\sigma(F)$, $G^\theta(F)$ and $G_\delta(F)$, respectively. We can also define stable versions:

$$SO_{\gamma_0}(\phi) = \sum_{\gamma_0 \sim \gamma} e(H_\gamma)O_\gamma(\phi)$$

$$STO_{\delta_0}(f) = \sum_{\delta_0 \sim \delta} e(G_\delta)TO_\delta(f),$$

where both sums are taken over a set of representatives for the relative classes (resp. twisted relative classes) inside the stable relative class (resp. stable twisted relative class). Here, the constants $e(H_\gamma)$ and $e(G_\delta)$ are the *Kottwitz signs*, as defined in (Lab99, §1.7). We note that if δ is relatively τ-regular semisimple (resp. γ is relatively regular semisimple), then G_δ (resp. H_γ) is a torus, and so its Kottwitz sign is 1.

4.3 Relative Fundamental Lemma

Recall that G and H are unramified groups. Therefore, they both admit smooth connected models over $\mathcal{O}_F$ (cf. § 1.1.1). We will still denote these models by G and H, respectively. That is, both G and H will henceforth be construed as group schemes over $\mathcal{O}_F$. As a first step towards a future trace comparison, we want to prove a *relative fundamental lemma*:

Conjecture 4.3.1. *Let $\delta \in G(F)$ be relatively τ-regular semisimple and $\gamma \in H(F)$ be relatively regular semisimple. Suppose γ is a norm for δ. Then*

$$STO_\delta(\mathbf{1}_{G(\mathcal{O}_F)}) = SO_\gamma(\mathbf{1}_{H(\mathcal{O}_F)}),$$

where the implicit measures give volume 1 to both $G(\mathcal{O}_F)$ and $H(\mathcal{O}_F)$ and are compatible with the inner twist $G_\delta \cong H_\gamma$ in the sense of Kottwitz (Kot88, p.631). Also, both $STO_\delta(\mathbf{1}_{G(\mathcal{O}_F)})$ and $SO_\gamma(\mathbf{1}_{H(\mathcal{O}_F)})$ vanish whenever δ does not admit a norm and γ is not a norm, respectively.

We will not prove the fundamental lemma in full generality, but by restricting δ and γ, we will prove a weaker version (cf. Theorem 4.3.7).

Recall that the matching of the geometric sides of the trace formula can be done locally. At unramified places, we want to choose test functions on our groups in a functorial manner: we want this correspondence to be a morphism of Hecke algebras. In these Hecke algebras, the unit element is given by the characteristic function of our hyperspecial subgroup, and therefore we want $\mathbf{1}_{G(\mathcal{O}_F)}$ to correspond $\mathbf{1}_{H(\mathcal{O}_F)}$. In the absolute case, Waldspurger proved that the fundamental lemma (i.e. the "matching" of unit elements) implies a similar matching statement for *all* test functions in the Hecke algebra; moreover, this special case had already been proven by Kottwitz in the case of base change (Kot86). Unfortunately, such a result is currently unavailable in the relative setting.

Since we are working with characteristic functions, the relative orbital integrals simplify considerably. Let $K = G(\mathcal{O}_F)$ and $K_H = H(\mathcal{O}_F)$. These hyperspecial subgroups have the following useful properties:

1. The map

$$K_H \to Q(F) \cap K_H$$
$$k \mapsto kk^{-\sigma}$$

   is surjective.

2. The maps

$$K \to Q(M) \cap K$$
$$k \mapsto kk^{-\sigma}$$

   and

$$K \to S(F) \cap K$$
$$k \mapsto kk^{-\theta}$$

   are surjective.

This follows from the fact that the fibers are (flat) torsors over the hyperspecial model. Hence, by Lemma 1.1.6, the fibers are non-empty. We will use these properties in the proof of the following lemma.

Lemma 4.3.2. *Set $K^\sigma = K \cap G^\sigma(F)$ and $K_H^\sigma = K_H \cap H^\sigma(F)$. Then*

$$O_\gamma(\mathbf{1}_{K_H}) = \sum_{y = hK_H^\sigma} \mathrm{vol}(C_{\gamma\gamma^{-\sigma}, H^\sigma}(F)_y)^{-1},$$

where the sum is over the $C_{\gamma\gamma^{-\sigma}, H^\sigma}(F)$-orbits in $X_H^\sigma := H^\sigma(F)/K_H^\sigma$ satisfying $h^{-1}\gamma\gamma^{-\sigma}h \in K_H$ and $C_{\gamma\gamma^{-\sigma}, H^\sigma}(F)_y$ is the stabilizer of y in $C_{\gamma\gamma^{-\sigma}, H^\sigma}(F)$.

Similarly, we have

$$TO_\delta(\mathbf{1}_K) = \sum_{x=gK^\sigma} \mathrm{vol}(C^\tau_{\delta\delta^{-\theta},G^\sigma}(F)_x)^{-1},$$

where the sum is over the $C^\tau_{\delta\delta^{-\theta},G^\sigma}(F)$*-orbits in* $X^\sigma := G^\sigma(F)/K^\sigma$ *satisfying* $g^{-1}\delta\delta^{-\theta}g^\tau \in K$ *and* $C^\tau_{\delta\delta^{-\theta},G^\sigma}(F)_x$ *is the stabilizer of* x *in* $C^\tau_{\delta\delta^{-\theta},G^\sigma}(F)$*. In both cases, the Haar measures on* $G^\sigma(F)$ *and* $H^\sigma(F)$ *are normalized to give* K^σ *and* K^σ_H *unit volume, and the volumes are taken with respect to fixed Haar measures on* $G_\delta(F)$ *and* $H_\gamma(F)$*.*

Proof. We will only prove the second equation: the first one will follow by choosing τ to be trivial.

Assume $\delta \in G(F)$ is relatively τ-semisimple. Note that we can partition $G_\delta(F)\backslash G^\sigma \times G^\theta(F)$ in terms of orbits for the (right) action of $K^\sigma \times K^\theta$, where $K^\theta = K \cap G^\theta(F)$. Therefore, we have

$$TO_\delta(\mathbf{1}_K) = \sum_{u,h} \mathrm{vol}\left(G_\delta(F)\backslash G_\delta(F)(uK^\sigma, hK^\theta)\right),$$

where the sum is over $u \in G^\sigma(F)$, $h \in G^\theta(F)$ such that $u^{-1}\delta h \in K$. Using Equation 4.1.0.2, we deduce that this sum is also equal to

$$\sum_x \mathrm{vol}\left(G_\delta(F)_x\right)^{-1},$$

where the sum is over a set of representatives for the G_δ-orbits of $x = (uK^\sigma, hK^\theta)$ such that $u^{-1}\delta h \in K$. Also, $G_\delta(F)_x$ denotes the stabilizer of x.

Now consider the following map

$$\left\{G_\delta(F)(uK^\sigma, hK^\theta) \in G_\delta(F)\backslash G^\sigma \times G^\theta(F)/K^\sigma \times K^\theta : u^{-1}\delta h \in K\right\}$$

$$\downarrow$$

$$\left\{C^\tau_{\delta\delta^{-\theta},G^\sigma}(F)uK^\sigma \in C^\tau_{\delta\delta^{-\theta},G^\sigma}(F)\backslash G^\sigma(F)/K^\sigma : u^{-1}\delta\delta^{-\theta}u^\tau \in K\right\}, \qquad (4.3.0.3)$$

defined by $(uK^\sigma, hK^\theta) \mapsto uK^\sigma$. By Lemma 3.1.2, this is well-defined. We claim it is also bijective. First, note that if $u^{-1}\delta h \in K$ then $hK = \delta^{-1}uK$, and since δ is fixed, we see that the coset of h determines the coset of u, and vice-versa. Moreover, if $hK = h'K$, then $h^{-1}h' \in K^\theta$ (since $h, h' \in G^\theta(F)$). Therefore, hK^θ is also determined by uK and hence by uK^σ. Therefore, the map 4.3.0.3 is injective. To prove surjectivity, we first note that whenever $gg^{-\theta} \in K$, properties (1) and (2) above imply that we can find $k \in K$ such that $gg^{-\theta} = kk^{-\theta}$. In particular, we see that $g \in KG^\theta(F)$. With this in mind, let $u \in G^\sigma(F)$ be such that $u^{-1}\delta\delta^{-\theta}u^\tau \in K$. Since $\delta^{-\theta}u^\tau = (u^{-1}\delta)^{-\theta}$, it follows that $u^{-1}\delta \in KG^\theta(F)$, and therefore there exists $h \in G^\theta(F)$ such that $u^{-1}\delta h \in K$, which proves surjectivity.

By Lemma 3.1.2, we have an isomorphism $G_\delta \cong C^\tau_{\delta\delta^{-\theta},G^\sigma}$, which we can use to transport our measure on $G_\delta(F)$ to a measure on $C^\tau_{\delta\delta^{-\theta},G^\sigma}(F)$. Using this measure and the equality

$$TO_\delta(\mathbf{1}_K) = \sum_{x=(uK^\sigma,hK^\theta)} \operatorname{vol}\left(G_\delta(F)_x\right)^{-1},$$

the bijection 4.3.0.3 gives us the desired result.

□

Remark. As we noted in §1.1 of Chapter 3, if γ is a norm of δ, their centralizers are inner forms of one another. In the statement of the fundamental lemma, one requires that the choice of measures on these centralizers correspond under the relevant inner twist (as mentioned above). However, if we restrict ourselves to relatively regular semisimple elements, the centralizers are actually tori, and this inner twist is therefore trivial, i.e. the centralizers are isomorphic.

We will need the following notation: set

$$V_\delta^\tau(G) = \left\{u \in G^\sigma(F) : u^{-1}\delta\delta^{-\theta}u^\tau \in K\right\}$$

and

$$V_\gamma(H) = \left\{h \in H^\sigma(F) : h^{-1}\gamma\gamma^{-\sigma}h \in K_H\right\}.$$

Consider now the following cohomological result.

Lemma 4.3.3. *We have* $H^1(\tau, K) = 1$. *In particular,* $(G(F)/K)^\tau = H(F)/K_H$.

Proof. By (Kot80, Lemma 8.6), we have $H^1(\mathrm{Gal}(M/F), K) = 1$. Now, consider two different structures of $\langle\tau\rangle$-group on K: the one coming from the definition of τ, and another one where τ acts as the non-trivial element of $\mathrm{Gal}(M/F)$. We then have an isomorphism of $\langle\tau\rangle$-groups

$$K \to K, \quad k \mapsto \tau(\overline{k}).$$

The result now follows from Kottwitz's lemma; the final assertion is proven by considering the long exact sequence associated to following short exact sequence:

$$1 \to K \to G(F) \to G(F)/K \to 1.$$

□

Therefore, the fixed points of $\gamma \in H(F)$ on $H(F)/K_H$ are exactly its fixed points on $G(F)/K$ which are also fixed by τ.

From the properties of hyperspecial groups preceding Lemma 4.3.2, we deduce that the image of the natural (injective) maps

$$G^\sigma(F)/K^\sigma \to G(F)/K, \qquad H^\sigma(F)/K_H^\sigma \to H(F)/K_H$$

are exactly the fixed points of σ. This gives us the following picture:

$$\begin{array}{ccc} H^\sigma(F)/K_H^\sigma & \longrightarrow & G^\sigma(F)/K^\sigma \\ \downarrow & & \downarrow \\ H(F)/K_H & \longrightarrow & G(F)/K \end{array},$$

where all maps are injective, the image of the horizontal maps is the fixed points of τ and the image of the vertical maps, the fixed points of σ. Therefore, the fixed points of $\gamma \in H(F)$ on $H^\sigma(F)/K_H^\sigma$ are exactly its fixed points on $H(F)/K_H$ which are also fixed by σ, and similarly for $\delta \in G(F)$. This picture suggests that the computations involved in the proof of the fundamental lemma should be combinatorial in nature: all these fixed points live in the (nonreduced) building of GL_n. This allows for more structure on the set of fixed points (since buildings are simplicial complexes). Langlands also made the same observation in the absolute case. However, the complexity of the computations grows rapidly with the rank of G, and a proof of the relative fundamental lemma for all n will therefore necessitate a different approach.

To better understand the computations required to prove the fundamental lemma, we make the following observation.

Lemma 4.3.4. *The relative orbital integrals only depend on the relative class, and similarly for the twisted case.*

Proof. We have a bijection between the fixed points, and the centralizers (and stabilizers) are conjugate. Our groups being unimodular, conjugation leaves the volumes invariant. Alternatively, replacing γ or δ by an element of their relative class amounts to a change of variable for the relative orbital integral. $\square$

If $TO_\delta(\mathbf{1}_{G(\mathcal{O}_F)}) \neq 0$, the twisted relative class of δ and K intersect non-trivially, and similarly for γ. Therefore, we can and do assume that $\delta \in K$ and $\gamma \in K_H$. In the absolute case the complexity of the fixed point sets $V_\delta^\tau(G)$ and $V_\gamma(H)$ should be related to the congruence relations between the eigenvalues (see for example (Kot05, §5)). This observations leads us to the following definition.

Definition 4.3.5. Let $\delta \in K$ be relatively τ-regular semisimple. The *depth* of δ is the least integer k such that the image of $\delta\delta^{-\theta}(\delta\delta^{-\theta})^\tau$ in $G(\mathcal{O}_F/\mathfrak{m}^{k+1})$ is regular, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_F$. Similary, if $\gamma \in K_H$ is relatively regular semisimple, its *depth* is the least integer k such that the image of $\gamma\gamma^{-\sigma}$ in $H(\mathcal{O}_F/\mathfrak{m}^{k+1})$ is regular.

In other words, the "depth zero" case alluded to in the introduction to this chapter is the case where δ (resp. γ) is still relatively τ-regular semisimple (resp. relatively regular semisimple) after reduction modulo $\mathfrak{m}$. Furthermore, we note that the depth is an invariant of the stable relative class.

Proposition 4.3.6. *Let $\delta \in K$ be relatively τ-regular semisimple, and let $\gamma \in K_H$ be relatively regular semisimple. Assume they both have depth zero. Then, we have*

$$TO_\delta(\mathbf{1}_{G(\mathcal{O}_F)}) = 1 = O_\gamma(\mathbf{1}_{H(\mathcal{O}_F)}).$$

From this proposition follows our main theorem:

Theorem 4.3.7. *The relative fundamental lemma holds in the depth zero case.*

Proof. Let δ and γ be relatively τ-regular semisimple (resp. relatively regular semisimple). If we suppose that γ is a norm of δ, then it follows that δ and γ have the same depth. Therefore, the theorem follows directly from the previous proposition. □

The proof of Proposition 4.3.6 will take the remainder of this section. The strategy is the following: first we compute the fixed point set of $g \in \mathrm{GL}_4(M)$ on $G(F)/K$ (as we would in the absolute case), and then we relate this computation to the relative setting.

Lemma 4.3.8. *Assume $g \in \mathrm{GL}_4(M)$ is regular semisimple of depth zero. Suppose also that g is σ-split. Then its fixed point set on $G^\sigma(F)/K^\sigma$ is $C_{g,G^\sigma}(F)/(C_{g,G^\sigma}(F) \cap K^\sigma)$.*

Proof. First, assume $g \in \mathrm{GL}_4(M)$ is diagonal. We will show that its fixed point set on $G(F)/K$ is $A/(A \cap K)$, where A is the subgroup of diagonal matrices. By the Iwasawa decomposition, we have $G(F) = ANK$, where N is the subgroup of unipotent matrices. Therefore, it suffices to show that if $n \in N$ and $g(nK) = nK$, then $n \in K$. We also observe that the hypothesis on the depth has the following consequence: if $\lambda, \mu \in M$ are eigenvalues of g, then $\mathrm{val}(\lambda - \mu) = 0$.

We write

$$n = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \qquad g = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{pmatrix},$$

where $a, b, d \in \mathrm{GL}_2(M)$, and a, d are unipotent. Suppose $g(nK) = nK$. The group of unipotent matrices in $\mathrm{GL}_2(M)$ can be canonically identified with M; let ρ be this canonical group isomorphism. By computing $n^{-1}gn$, we deduce that in order for $g(nK) = nK$ we need

$$\rho(a)\left(\lambda_1 - \lambda_2\right), \rho(d)\left(\lambda_3 - \lambda_4\right) \in \mathcal{O}_M.$$

It follows that $\rho(a), \rho(d) \in \mathcal{O}_M$, and we can therefore assume for the remaining computations that $a = d = I_2$. Write

$$b = \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Again, by computing $n^{-1}gn$, we deduce that in order for $g(nK) = nK$ we need

$$\begin{pmatrix} x(\lambda_1 - \lambda_3) & y(\lambda_1 - \lambda_4) \\ z(\lambda_2 - \lambda_3) & w(\lambda_2 - \lambda_4) \end{pmatrix} \in \mathrm{Mat}_2(\mathcal{O}_M).$$

As above, we can conclude that $b \in \mathrm{Mat}_2(\mathcal{O}_M)$, which is what we wanted to show.

Now, if g is hyperbolic (i.e. its eigenvalues lie in M) but not diagonal, we can still conclude that its fixed point set on $G(F)/K$ is $C_{g,G}(F)/\left(C_{g,G}(F) \cap K\right)$: some conjugate of g is diagonal, and conjugation induces a bijection between fixed-point sets. Since g is σ-split, its centralizer $C_{g,G}$ is stable under σ, and so the fixed point set of g on $G^\sigma(F)/K^\sigma$ is itself preserved by $C_{g,G}(F)$. The result thus follows for this case.

Finally, suppose g is not hyperbolic, and let L/M be an extension over which g becomes hyperbolic. Note that this extension has degree 2 or 4, and therefore it has at most tame ramification (recall the hypothesis on the residual characteristic). By (Tit79, Proposition 2.6.1), the fixed point set of g on $G(F)/K$ is exactly the set of its fixed points on $\mathrm{GL}_4(L)/\mathrm{GL}_4(\mathcal{O}_L)$ which are also $\mathrm{Gal}(L/M)$-invariant. By the reasoning above, we know that

$$(\mathrm{GL}_4(L)/\mathrm{GL}_4(\mathcal{O}_L))^g = C_{g,\mathrm{GL}_4}(L)/\left(C_{g,\mathrm{GL}_4}(L) \cap \mathrm{GL}_4(\mathcal{O}_L)\right),$$

and by (Kot80, Lemma 8.6), we can now deduce that

$$(G(F)/K)^g = C_{g,G}(F)/(C_{g,G}(F) \cap K),$$

and the result now follows.

□

We note that the action of $C_{g,G^\sigma}(F)$ on $C_{g,G^\sigma}(F)/(C_{g,G^\sigma}(F) \cap K^\sigma)$ is transitive. Moreover, the stabilizer of $C_{g,G^\sigma}(F) \cap K^\sigma$ is of course $C_{g,G^\sigma}(F) \cap K^\sigma$, and we also have $\mathrm{vol}(C_{g,G^\sigma}(F) \cap K^\sigma) = 1$.

Proof of Proposition 4.3.6. To finish the proof, we simply have to consider the following incarnations for g: $g = \gamma\gamma^{-\sigma}$ and $g = \delta\delta^{-\theta}(\delta\delta^{-\theta})^\tau$. In the first case, Lemma 4.3.8

and (Kot80, Lemma 8.6) readily imply that

$$V_\gamma(H) = C_{\gamma\gamma^{-\sigma},H^\sigma}(F)/(C_{\gamma\gamma^{-\sigma},H^\sigma}(F) \cap K_H^\sigma),$$

and we thus have

$$O_\gamma(\mathbf{1}_{K_H}) = \mathrm{vol}(C_{\gamma\gamma^{-\sigma},H^\sigma}(F) \cap K_H^\sigma) = 1.$$

Finally, if we apply the twisted action of $\delta\delta^{-\theta}$ twice, we see that

$$V_\delta^\tau(G) \subset (G^\sigma(F)/K^\sigma)^{\delta\delta^{-\theta}(\delta\delta^{-\theta})^\tau}.$$

As above, we thus have

$$TO_\delta(\mathbf{1}_K) = \mathrm{vol}(C^\tau_{\delta\delta^{-\theta},G^\sigma}(F) \cap K^\sigma) = 1.$$

□

Unfortunately, even though the proof above suggests that proving the relative fundamental lemma should simply follow from a computation in the absolute setting, the combinatorial nature and the complexity of the computations seem prohibitive at the moment.

References

[AC89] J. Arthur and L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Mathematics Studies, vol. 120, Princeton University Press, 1989.

[Art97] J. Arthur, *Stability and endoscopy: Informal motivation*, Representation Theory and Automorphic Forms (T. Bailey and A. Knapp, eds.), Proceedings of Symposia in Pure Mathematics, no. 61, 1997.

[Art05] ———, *An introduction to the trace formula*, Harmonic Analysis, the Trace Formula, and Shimura Varieties (J. Arthur, D. Ellwood, and Kottwitz R., eds.), American Mathematical Society, 2005.

[Art13] ———, *The endoscopic classification of representations: Orthogonal and symplectic groups*, Colloquium Publications, American Mathematical Society, 2013.

[BL84] J.-L. Brylinski and J.-P. Labesse, *Cohomologie d'intersection et fonctions L de certaines variétés de Shimura*, Annales scientifiques de l'E.N.S. **17** (1984), 361–412.

[BLR90] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer, 1990.

[BR92] D. Blasius and J. Rogawski, *Fundamental lemmas for U(3) and related groups*, The Zeta Functions of Picard Modular Surfaces (R. Langlands and D. Ramakrishnan, eds.), Les Publications CRM, 1992.

[BR94] ———, *Zeta-functions of Shimura varieties*, Motives (U. Jannsen, S. Kleiman, and J.-P. Serre, eds.), Proceedings of Symposia in Pure Mathematics, no. 55, 1994.

[CGP10] B. Conrad, O. Gabber, and G. Prasad, *Pseudo-reductive groups*, New Mathematical Monographs, Cambridge University Press, 2010.

[Cog04] J. Cogdell, *Dual groups and Langlands functoriality*, An Introduction to the Langlands' Program (J. Bernstein and S. Gelbart, eds.), Birkhauser, 2004.

[Edi92] B. Edixhoven, *Néron models and tame ramification*, Compositio Mathematica **81** (1992), 291–306.

[EH00] D. Eisenbud and J. Harris, *Geometry of schemes*, Graduate Texts in Mathematics, Springer, 2000.

[FJ93] S. Friedberg and H. Jacquet, *Linear periods*, Journal für die reine und angewandte Mathematik **443** (1993), 91–139.

[Gel96] S. Gelbart, *Lectures on the Arthur-Selberg trace formula*, University Lecture Series, vol. 9, American Mathematical Society, 1996.

[Get] J. Getz, *A relative fundamental lemma with an application to unitary groups*, Unpublished.

[Get12] ———, *An approach to nonsolvable base change and descent*, Journal of the Ramanujan Mathematical Society **27** (2012), 143–211.

[GGPS69] I. Gelfand, M. Graev, and I. Piatetski-Shapiro, *Representation theory and automorphic functions*, W.B. Saunders Company, 1969.

[GW] J. Getz and E. Wambach, *Twisted relative trace formulae with a view towards unitary groups*, To appear in the American Journal of Mathematics.

[Hah09] H. Hahn, *A simple twisted relative trace formula*, International Mathematics Research Notices **2009** (2009), 3957–3978.

[Har11] M. Harris, *An introduction to the stable trace formula*, On the Stabilization of the Trace Formula (L. Clozel, M. Harris, J.-P. Labesse, and B.-C. Ngo, eds.), Stabilization of the Trace Formula, Shimura Varieties, and Arithmetic Applications, no. 1, 2011.

[Hel91] A. Helminck, *Tori invariant under an involutorial automorphism i*, Advances in Mathematics **85** (1991), 1–38.

[HL04] M. Harris and J.-P. Labesse, *Conditional base change for unitary groups*, Asian Journal of Mathematics **8** (2004), 653–684.

[HLR86] G. Harder, R. Langlands, and M. Rapoport, *Algebraische Zyklen auf Hilbert-Blumenthal-Flächen*, Journal für die reine und angewandte Mathematik **366** (1986), 53–120.

[Hum75] J. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics, Springer-Verlag, 1975.

[HZ76] F. Hirzebruch and D. Zagier, *Intersection numbers of curves on Hilbert modular surfaces and modular forms with nebentypus*, Inventiones Mathematicae **36** (1976), 57–113.

[Jac97] H. Jacquet, *Automorphic spectrum of symmetric spaces*, Representation Theory and Automorphic Forms (T. Bailey and A. Knapp, eds.), Proceedings of Symposia in Pure Mathematics, no. 61, 1997.

[Jac10] ______, *Distinction by the quasi-split unitary group*, Israel Journal of Mathematics **178** (2010), 269–324.

[JL85] H. Jacquet and K. Lai, *A relative trace formula*, Compositio Mathematica **54** (1985), 243–310.

[JR96] H. Jacquet and S. Rallis, *Uniqueness of linear periods*, Compositio Mathematica **102** (1996), 65–123.

[Kot80] R. Kottwitz, *Orbital integrals on GL(3)*, American Journal of Mathematics **102** (1980), 327–384.

[Kot82] ______, *Rational conjugacy classes in reductive groups*, Duke Mathematical Journal **49** (1982), 785–806.

[Kot86] ______, *Base change for unit elements of Hecke algebras*, Compositio Mathematica **60** (1986), 237–250.

[Kot88] ______, *Tamagawa numbers*, Annals of Mathematics **127** (1988), 629–646.

[Kot05] ______, *Harmonic analysis on reductive p-adic groups and Lie algebras*, Harmonic Analysis, the Trace Formula, and Shimura Varieties (J. Arthur, D. Ellwood, and R. Kottwitz, eds.), American Mathematical Society, 2005.

[Lab99] J.-P. Labesse, *Cohomologie, stabilisation et changement de base*, Astérisque, vol. 257, Société Mathématique de France, 1999.

[Lai85] K.F. Lai, *Algebraic cycles on compact Shimura surface*, Mathematische Zeitschrift **189** (1985), 593–602.

[Lan56] S. Lang, *Algebraic groups over finite fields*, American Journal of Mathematics **78** (1956), 555–563.

[Lan80] R. Langlands, *Base change for GL(2)*, Annals of Mathematics Studies, Princeton University Press, 1980.

[Mil80] J. Milne, *Étale cohomology*, Princeton University Press, 1980.

[Mil92] ———, *The points on a Shimura variety modulo a prime of good reduction*, The Zeta Functions of Picard Modular Surfaces (R. Langlands and D. Ramakrishnan, eds.), Les Publications CRM, 1992.

[Mok] C.P. Mok, *Endoscopic classification of representations of quasi-split unitary groups*, Preprint.

[PR94] V. Platonov and A. Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, Academic Press, 1994.

[Ric82] R. Richardson, *Orbits, invariants, and representations associated to involutions of reductive groups*, Inventiones mathematicae **66** (1982), 287–312.

[Rog90] J. Rogawski, *Automorphic representations of unitary groups in three variables*, Annals of Mathematics Studies, Princeton University Press, 1990.

[Ser68] J.-P. Serre, *Corps locaux*, Actualités Scientifiques et Industrielles, Hermann, 1968.

[Ser73] J.P. Serre, *A course in arithmetic*, Graduate Texts in Mathematics, Springer, 1973.

[Ser97] J.-P. Serre, *Cohomologie galoisienne*, Lectures Notes in Mathematics, Springer, 1997.

[Spr98] T. Springer, *Linear algebraic groups*, Birkhaüser, 1998.

[Str83] R. Strano, *Principal homogeneous spaces over Hensel rings*, Proceedings of the American Mathematical Society **87** (1983), 208–212.

[Tat65] J. Tate, *Algebraic cycles and poles of zeta functions*, Arithmetical Algebraic Geometry (O. F. G. Schilling, ed.), Harper and Row, 1965.

[Tat79] ______, *Number-theoretic background*, Automorphic Forms, Representations, and L-functions (A. Borel and W. Casselman, eds.), Proceedings of Symposia in Pure Mathematics, no. 33, 1979.

[Tit79] J. Tits, *Reductive groups over local fields*, Automorphic Forms, Representations, and L-functions (A. Borel and W. Casselman, eds.), Proceedings of Symposia in Pure Mathematics, no. 33, 1979.

[Yu] J.K. Yu, *Smooth models associated to concave functions in Bruhat-Tits theory*, Preprint.